COMP/MATH 3804 – Design and Analysis of Algorithms I

Fall 2015

Assignment 6

due on December 02, 2015

Problem 1. Consider a sequence of n distinct integers, $S = (a_1, a_2, \cdots, a_n)$. A *bad pair* of S is a pair of indices (i, j) such that $i < j$ and $a_i \geq 2 \cdot a_j$. For instance, the sequence $(6, 2, 5, 8, 3, 4)$ has four bad pairs: $(1, 2)$, $(1, 5)$, $(4, 5)$, and $(4, 6)$. Describe (in English) a divide-and-conquer algorithm that counts the number of bad pairs in a given sequence. The algorithm should run in time $O(n \log n)$. *All pseudo-code will be ignored.*

Problem 2. Let a multiset of n integers $S = \{a_1, a_2, \cdots, a_n\}$ be given, where $0 \leq a_i \leq n$ for all $1 \leq i \leq n$. Use dynamic programming to solve the following problem in time $O(n^3)$: find a partition $S = S_1 \cup S_2$ that minimizes the function

$$\left| \sum_{i \in S_1} a_i - \sum_{j \in S_2} a_j \right|.$$

All pseudo-code will be ignored.

Problem 3. You are given a sequence of integers $(d_1, d_2, \cdots, d_n)$. Describe (in English) an efficient greedy algorithm that constructs an undirected graph on n vertices such that, for all $1 \leq i \leq n$, the i-th vertex has exactly d_i neighbors, or reports that such a graph does not exist. *All pseudo-code will be ignored.*

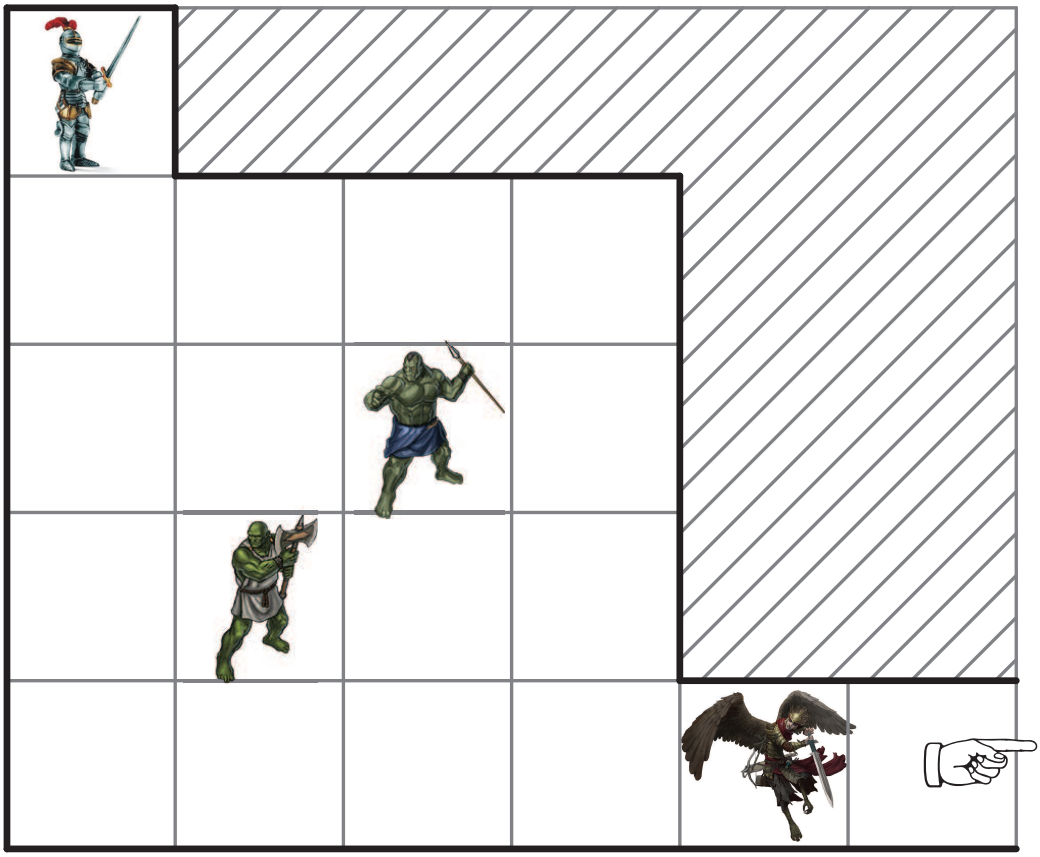
Problem 4. A knight has to fight his way out of a maze infested by hideous monsters. The maze consists of an $m \times n$ matrix of square *cells*, each of which can contain either *nothing*, or a *wall*, or a *monster*, or the *knight*. The knight is initially located in $(1, 1)$, and the exit is in (m, n). The knight can freely move to adjacent empty cells, but he cannot go through walls. Monsters instead are stationary guards, and hence they never move, nor do they attack the knight, unless he invades the cells they occupy.

The knight can walk into a monster, provided that he is equipping the right weapon to fight that monster. There are several types of monsters and several types of weapons, and some weapons are ineffective against some monsters. If the knight is carrying the proper weapon, he can slay a monster upon walking onto its cell. Every time the knight slays a monster, he loses one *health point* in the fight, and the monster is permanently eliminated. Monsters may carry weapons too, and the knight can pick up weapons dropped by slain monsters, and use them in later fights. Any monster can carry any type of weapon (note that the types of weapons that can defeat a monster are determined only by the monster's type, and not by the type of weapon that the monster is carrying). The knight starts off with just a *sword*, and can carry all the weapons that he finds. Weapons have infinite durability, but the knight has only h initial health points, and no way to recover them. If the knight's health points drop to 0, the knight dies. If the knight walks into a monster without equipping an appropriate weapon, he is instantly killed.

The knight has a detailed map of the maze, including the locations of all monsters and the weapons they are carrying. He also knows his current location on the map, as well as the location of the exit. So, he can carefully plan out his strategy before taking action. His task is to find a sequence of moves that will allow him to exit the maze alive, or die a noble death if he determines that his doom is inevitable.

Prove that the knight's problem is NP-complete. (In your NP-hardness reduction, you can build mazes of any size, with any number of monsters, monster types and weapon types, and you can freely decide which weapons kill which monsters, and what the initial health points of the knight are.)

Example. Say that the knight has initially $h = 3$ health points, and that there are two types of monsters in the maze: *orcs*, who can be slain by any weapon, and *harpies*, who can be slain only by *javelins*. The map of the maze is the following:



The knight starts in the top-left corner, carrying a sword, and there are two orcs in the main room, carrying an axe and a javelin, respectively. There is also a harpy carrying a sword, who is blocking the way out of the maze.

The knight can immediately slay any orc with his sword, or he can decide to walk around them and let them live. However, he must slay the harpy, because it is blocking the exit. In order to slay the harpy he needs a javelin, and to get the javelin he has to slay the orc carrying it. Note that he does not have to slay the other orc, because getting his axe would be useless. In fact, slaying both orcs would be a deadly mistake, because then the knight's health points would drop to zero during the final fight against the harpy.

Therefore there is only one correct strategy for the knight, which is:

- Walk to the orc who carries the javelin.

- Slay this orc with the sword and pick up the javelin.
- Avoid the other orc, and walk to the harpy.
- Equip the javelin and slay the harpy.
- Ignore the sword dropped by the harpy and walk to the exit.