Distributed Computing by Mobile Robots: Solving the Uniform Circle Formation Problem

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Abstract

Consider a set of $n \neq 4$ simple autonomous mobile robots (decentralized, asynchronous, no common coordinate system, no identities, no central coordination, no direct communication, no memory of the past, deterministic) initially in distinct locations, moving freely in the plane and able to sense the positions of the other robots. We study the primitive task of the robots arranging themselves equally spaced along a circle not fixed in advance (UNIFORM CIRCLE FORMATION). In the literature, the existing algorithmic contributions are limited to restricted sets of initial configurations of the robots and to more powerful robots. The question of whether such simple robots could deterministically form a uniform circle has remained open. In this paper, we constructively prove that indeed the UNIFORM CIRCLE FORMATION problem is solvable for any initial configuration of the robots without any additional assumption. In addition to closing a long-standing problem, the result of this paper also implies that, for pattern formation, asynchrony is not a computational handicap, and that additional powers such as chirality and rigidity are computationally irrelevant.

1 Introduction

Consider a set of punctiform computational entities, called *robots*, located in $\mathbb{R}^2$ where they can freely move. Each entity is provided with a local coordinate system and operates in *Look-Compute-Move* cycles. During a cycle, a robot obtains a snapshot of the positions of the other robots, expressed in its own coordinate system (*Look*); using the snapshot as an input, it executes an algorithm (the same for all robots) to determine a destination (*Compute*);[1] and it moves towards the computed destination (*Move*). After a cycle, a robot may be inactive for some time.

To understand the nature of the distributed universe of these mobile robots and to discover its computational boundaries, the research efforts have focused on the minimal capabilities the robots need to have to be able to solve a problem. Thus, the extensive literature

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[1] We assume that a robot can compute algebraic functions of the points in the input snapshot, with infinite precision.

on distributed computing by mobile robots has almost exclusively focused on very simple entities operating in strong adversarial conditions. The robots we consider are *anonymous* (without ids or distinguishable features), *autonomous* (without central or external control), *oblivious* (no recollection of computations and observations done in previous cycles), *asynchronous* (no synchronization among the robots' cycles nor their duration), *disoriented* (no agreement among the individual coordinate systems, nor on unit distance and chirality). In particular, the choice of individual coordinate systems, the activation schedule, the duration of each operation during a cycle, and the length traveled by a robot during its movement[2] are determined by an adversary; the only constraints on the adversary are fairness (i.e., for every time t and each robot r there exists $t' > t$ when r is active), finiteness (i.e., the duration of each activity and inactivity is arbitrary but finite), and minimality (i.e., there exists $\delta > 0$, unknown to the robots, such that if the destination is at distance at most δ the robot will reach it, else it will move at least δ towards the destination, and then it may be unpredictably stopped by the adversary). We will refer to this model as $\mathcal{ADO}$ (for the main characteristics of the robots: asynchronous, disoriented, and oblivious). The two other main models studied in the literature are the *semi-synchronous* and the *fully synchronous* models, denoted here as $\mathcal{SDO}$ and $\mathcal{FDO}$, respectively, where the robots, oblivious and disoriented, however operate in synchronous rounds, and each round is "atomic": all robots active in that round terminate their cycle by the next round; the only difference is whether all robots are activated in every round ($\mathcal{FDO}$), or, subject to some fairness condition, a possibly different subset is activated in each round ($\mathcal{SDO}$). All three models have been intensively studied (e.g., see [1, 2, 3, 5, 6, 7, 8, 9, 10, 14, 15, 16, 22, 23]; for a detailed overview refer to the recent monograph [13]).

The research on the *computability* aspects has focused almost exclusively on the fundamental class of Geometric Pattern Formation problems. A *geometric pattern* (or simply *pattern*) P is a set of points in the plane; the robots *form* the pattern P at time t if the configuration of the robots (i.e., the set of their positions) at time t is similar to P (i.e., coincident with P up to scaling, rotation, translation, and reflection). A pattern P is *formable* if there exists an algorithm that allows the robots to form P within finite time and no longer move, regardless of the activation scheduling and delays (which, recall, are decided by the adversary) and of the initial placement of the robots in distinct points. Given a model, the research questions are: to determine if a given pattern P is formable in that model; if so, to design an algorithm that will allow its formation; and, more in general, to fully characterize the set of patterns formable in that model. The research effort has focused on answering these questions for $\mathcal{ADO}$ and the less restrictive models both in general (e.g., [5, 14, 15, 21, 22, 23]) and for specific classes of patterns (e.g., [1, 7, 8, 10, 11, 12, 18, 19]).

Among specific patterns, a special research place is occupied by two classes: `Point` and `Uniform Circle`. The class `Point` is the set consisting of a single point; point formation corresponds to the important Gathering problem requiring all robots to gather at a same location, not determined in advance (e.g., see [2, 3, 4, 17, 20]). The other important class of patterns is `Uniform Circle`: the points of the pattern form the vertices of a regular n-gon,

[2] If the robots are guaranteed to always reach their destination point by the end of each *Move* phase, the model is said to be *rigid*. If, on the other hand, they can be stopped by the scheduler before they reach their destination point, the model is said to be *non-rigid*.

where n is the number of robots (e.g., [1, 6, 7, 8, 10, 11, 12, 19]).

In addition to their relevance as individual problems, the classes `Point` and `Uniform Circle` play another important role. A crucial observation, by Suzuki and Yamashita [22], is that formability of a pattern P from an initial configuration Γ in model $\mathcal{M}$ depends on the relationship between $\rho_{\mathcal{M}}(P)$ and $\rho_{\mathcal{M}}(\Gamma)$, where $\rho_{\mathcal{M}}(V)$ is a special parameter, called *symmetricity*, of a multiset of points V, interpreted as robots modeled by $\mathcal{M}$. Based on this observation, it follows that the only patterns that *might* be formable from any initial configuration in $\mathcal{FDO}$ (and thus also in $\mathcal{SDO}$ and $\mathcal{ADO}$) are single points and uniform circles.

It is rather easy to see that both points and uniform circles can be formed in $\mathcal{FDO}$, i.e. if the robots are fully synchronous. After a long quest by several researchers, it has been shown that Gathering is solvable (and thus `Point` is formable) in $\mathcal{ADO}$ (and thus also in $\mathcal{SDO}$) [2], leaving open only the question of whether `Uniform Circle` is formable in these models.

In $\mathcal{SDO}$, it was known that the robots can *converge* towards a uniform circle without ever forming it [7]. Some recent results indicate that the robots can actually form a uniform circle in $\mathcal{SDO}$. In fact, by concatenating the algorithm of [18], for forming a biangular configuration, with the one of [11], for circle formation from an equiangular starting configuration, it is possible[3] to form a uniform circle starting from any initial configuration in $\mathcal{SDO}$. Hence, the only outstanding question is whether it is possible to form a uniform circle in $\mathcal{ADO}$.

In spite of the simplicity of its formulation and the repeated efforts by several researchers, the existing algorithmic contributions are limited to restricted sets of initial configurations of the robots and to more powerful robots. In particular, it has been proven that, with the additional property of *chirality* (i.e., a common notion of "clockwise"), the robots can form a uniform circle [12], and with a very simple algorithm; the fact that `Uniform Circle` is formable in $\mathcal{ADO}$+*chirality* follows also from the recent general result of [15]. The difficulty of the problem stems from the fact that the inherent difficulties of asynchrony, obliviousness, and disorientation are amplified by their simultaneous presence.

In this paper we show that indeed the Uniform Circle Formation problem is solvable for any initial configuration of $n \neq 4$ robots without any additional assumption, closing a problem open for over a decade. This result also implies that, for Geometric Pattern Formation problems, *asynchrony* is not a computational handicap, and that additional powers such as *chirality* and *rigidity* are computationally irrelevant.

2 Definitions

For a finite set $S \subset \mathbb{R}^2$ of $n > 2$ points, we define the *smallest enclosing circle*, or *SEC*, to be the circle of smallest radius such that every point of S lies on the circle or in its interior. For any S, SEC is easily proven to exist and to be unique. Three other circles will play a special role: these are concentric with SEC, and have radiuses that are 1/2, 1/3, and 1/4 the radius of SEC. They are denoted by *SEC/2*, *SEC/3*, and *SEC/4*, respectively.

The *angular distance*, with respect to point x, between two points p and q (distinct from

[3]Notice that the two algorithms can be concatenated only if the robots are semi-synchronous.

x) is the measure of the smallest angle between $\angle pxq$ and $\angle qxp$, and is denoted by $\theta_x(p, q)$. The *sector* defined by two points a and b is the locus of points c such that $\theta_x(a, c)+\theta_x(c, b) = \theta_x(a, b)$. Whenever x is not specified, it is assumed to be the center of the SEC of a well-understood set of points.

Given a finite set S, the positions of its points around some point $x \notin S$, taken clockwise, naturally induce a cyclic order on S. If several points of S lie on the same ray emanating from x, their relative order is induced by their distance from x, starting from the nearest point.

Let $p_0 \in S$ be any point, and let $p_i \in S$ be the $(i + 1)$-th point in the cyclic order around $x \notin S$, starting from p_0. Let $\alpha_x^{(i)} = \theta_x(p_i, p_{i+1})$, where the indices are taken modulo n. Then, $(\alpha_x^{(i)})_{0\leqslant i<n}$ is called the *angle sequence* induced by p_0. (Of course, depending on the choice of $p_0 \in S$, there may be at most n different angle sequences with respect to x.) Letting $\beta_x^{(i)} = \alpha_x^{(n-i)}$, for $0 \leqslant i < n$, we call $(\beta_x^{(i)})_{0\leqslant i<n}$ the *reverse angle sequence* induced by p_0. We let $\widetilde{\alpha}_x$ and $\widetilde{\beta}_x$ be, respectively, the lexicographically smallest angle sequence and the lexicographically smallest reverse angle sequence of S. Also, we denote by μ_x the lexicographically smallest between $\widetilde{\alpha}_x$ and $\widetilde{\beta}_x$, and by $\mu_x^{(i)}$ the i-th element of μ_x. If $p \in S$ is any point inducing μ_x as a clockwise or counterclockwise angle sequence, we say that p is a *lex-first* point of S (with respect to x), and we denote by $\mathcal{L}_1$ the set of all lex-first points. Let p be a lex-first point of S and suppose that μ_x is the clockwise (resp. counterclockwise) angle sequence induced by p. Let p' be the first point after p in the clockwise (resp. counterclockwise) order around x that is not collinear with x and p. Then, p' is said to be a *lex-second* point of S (with respect to x), and we denote by $\mathcal{L}_2$ the set of all lex-second points. If x is not specified, it is assumed to be the center of the SEC of S.

The following definitions apply whenever the symbols used are well defined, i.e., if and only if no point of S lies in the center of SEC. S is *co-radial* if $\mu^{(0)} = 0$. In a co-radial set, every two points at angular distance 0 are said to be *co-radial* with each other. The number of distinct clockwise angle sequences of S (with respect to the center of its SEC) is called the *period* of S. It is easy to verify that the period is always a divisor of n.

We will be distinguishing among different types of configurations, defined below (see also Figure 3). S is said to be *Equiangular* if its period is 1, *Biangular* if its period is 2, and *Aperiodic* if its period is n. In a *Biangular* set, any two points at angular distance $\mu^{(0)}$ are called *neighbors*, and any two points at angular distance $\mu^{(1)}$ are called *quasi-neighbors*. If a *Biangular* configuration is not co-radial, it is called *Simple biangular*. An *Aperiodic* configuration can be *Uni-aperiodic* if $\widetilde{\alpha} \neq \widetilde{\beta}$, and *Bi-periodic* if $\widetilde{\alpha} = \widetilde{\beta}$. A set S that is not *Aperiodic* is said to be *Uni-periodic* if $\widetilde{\alpha} \neq \widetilde{\beta}$, and *Bi-periodic* if $\widetilde{\alpha} = \widetilde{\beta}$. S is *Regular* if its points are the vertices of a regular n-gon.

We say that point $p \in S$ is *homologous* to point $q \in S$ if the angle sequence induced by p is equal to the angle sequence induced by q, or to its reverse. In particular, if it is equal to the angle sequence induced by q (and not necessarily to its reverse), p and q are said to be *analogous*. Homology and analogy are equivalence relations on S, and the equivalence classes that they induce on S are called *homology classes* and *analogy classes*, respectively. In a *Uni-periodic* set of period k, all homology classes are *Equiangular* sets of size n/k. In a *Bi-periodic* set of period k, each homology class is either a *Biangular* set of size $2n/k$, or an *Equiangular* set of size n/k or $2n/k$. In a *Uni-aperiodic* set, the homology classes consist of

Algorithm Uniform Circle Formation

Find first match of observed configuration:

1. **Regular:** Do nothing;
2. **Central:** Execute Central;
3. **Equiangular:** Execute Equiangular;
4. **Pre-regular:** Execute Pre-regular;
5. **Pre-equiangular:** Execute Pre-equiangular;
6. **Landmark-co-radial:** Execute Landmark-co-radial;
7. **Post-periodic:** Execute Post-periodic;
8. **Antipodal-referees:** Execute Antipodal-referees;
9. **Simple Biangular:** Execute Simple biangular;
10. **Periodic:** Execute Periodic;
11. **Post-aperiodic:** Execute Post-aperiodic;
12. **Aperiodic:** Execute Aperiodic;

Figure 1: The Uniform Circle Formation algorithm

one point; in a *Bi-aperiodic*, they consist of either one or two points.

S is said to be *Double-biangular* if it is *Bi-periodic* with period 4 and has exactly two homology classes.

S is *Pre-regular* if there exists a regular n-gon (called the *supporting polygon*) such that, for each pair of adjacent edges, one edge contains exactly two points of S (possibly on its endpoints), and the other edge's relative interior contains no point of S [8]. There is a natural correspondence between points of S and vertices of the supporting polygon: the *matching vertex* v of point $p \in S$ is such that v belongs to the edge containing p, and the segment vp contains no other point of S. If two points of S lie on a same edge of the supporting polygon, then they are said to be *companions*.

Finally, S is *Central* if one of its points lies at the center of SEC.

3 The Algorithm

3.1 High-Level Description

The general idea of the algorithm is that first some robots identify themselves as *referees* (in spite of anonymity) and maintain their role until they are the only ones not in their final position. The referees univocally determine special points, the *landmarks*, which, in turn, define a set of half-lines from the centre of SEC, the *targets*, partitioning the plane in n equal sectors. Each robot is assigned a different target. By positioning themselves on the targets, the robots reach an *Equiangular* configuration, and they ultimately form a uniform circle.

Algorithm Uniform Circle Formation (see Figure 1) consists of an ordered set of tests to determine the class of the current configuration; this determines what action is going to be taken by a robot in order to implement the general strategy described above. The universe of possible configurations is decomposed by the algorithm into several classes. Some of the classes (i.e., *Regular*, *Central*, *Equiangular*, *Pre-regular*, *Simple biangular*, *Periodic*, *Aperiodic*) have been defined in Section 2; the others will be defined in the following, along with the description of the corresponding actions. We stress that some configurations belong to more than one class, and so the order in which such classes are tested by the algorithm matters.

3.2 Basic Tools

The above high-level description gives an idea of the general *intended behavior* of the robots. Asynchrony and special configurations can easily make the algorithm deviate from this behavior. The rules and movements of the robots are carefully designed so to handle any deviation, and they are quite complex. In particular, two tools are employed: cautious moves and special circles.

Cautious Move. If a robot's movement can potentially create some configuration that would be treated by other observing robots in an inconsistent way (i.e., a configuration of a class tested *before* the current one by the algorithm), the rule will prescribe the robot to stop in the first point that might create it. We call these points *critical points*. Thus in some procedures of the algorithm, robots are specifically required to perform an operation called *cautious move*; this method is invoked when there is a set of robots that need to move on disjoint paths, each of which contains finitely many critical points. It is assumed that, as the robots move along their paths, the set of critical points does not change.

In a cautious move, first the set of critical points is expanded with a set of "auxiliary" critical points: if a robot has a critical point on its path, located at distance d from the endpoint of the path (where the distance is measured along the path itself), then each other robot whose path is not shorter than d acquires a new critical point at distance d from the end of its path. The last point along each robot's path is also taken as a critical point.

Then, each robot r whose remaining path is longest moves forward along its path by the greatest possible amount, with the following constraints:

- r's destination point must not be past the next critical point (auxiliary or not);
- if r is currently lying on a critical point (auxiliary or not), its destination point must be at most halfway toward the next critical point (auxiliary or not) along its path;
- if the remaining path of r has length d, and there is another robot whose remaining path is has length $d' < d$, then r's destination point must be at most d' away from the endpoint of r's path (in other words, robots do not "pass each other" in one turn).

On the other hand, the robots whose remaining path is not longest wait.

Special Circles. In the algorithm we use specific concentric circles: SEC, SEC/2, SEC/3, and SEC/4. This is done first of all to facilitate the recognition of the current configuration and coordinate the operations of the robots. For example, SEC/4 is used in *Periodic* while SEC/3 is used in *Aperiodic*. More importantly, these circles are used to *avoid* the accidental formation of certain configurations. In particular, as long as some robots are on or inside SEC/3, a *Pre-regular* configuration may never be formed. This is crucial in the proof of correctness of the algorithm.

3.3 The Initial Tests

The first four tests performed by the algorithm are the simplest ones. The algorithm first checks if a uniform circle has been formed; if so, no further action is taken. Otherwise, it checks if there is a robot at the centre of SEC. In this case, that robot moves, avoiding collisions, to become co-radial with the robots on one of the most populated radiuses, and stopping before SEC/4. This action (procedure CENTRAL) transforms the configuration in one of class *Aperiodic*. In the third test, the algorithm checks if the configuration is *Equiangular*; if so, all robots move radially towards SEC eventually evolving into a *Regular* configuration. In the fourth test, if the configuration is *Pre-regular*, each robot moves towards its matching vertex in the supporting polygon. This action, called procedure PRE-REGULAR is precisely the technique described in [8] to move from a *Biangular* configuration into a *Regular* one; during the action the configuration remains *Pre-regular* and it eventually evolves into *Regular*.

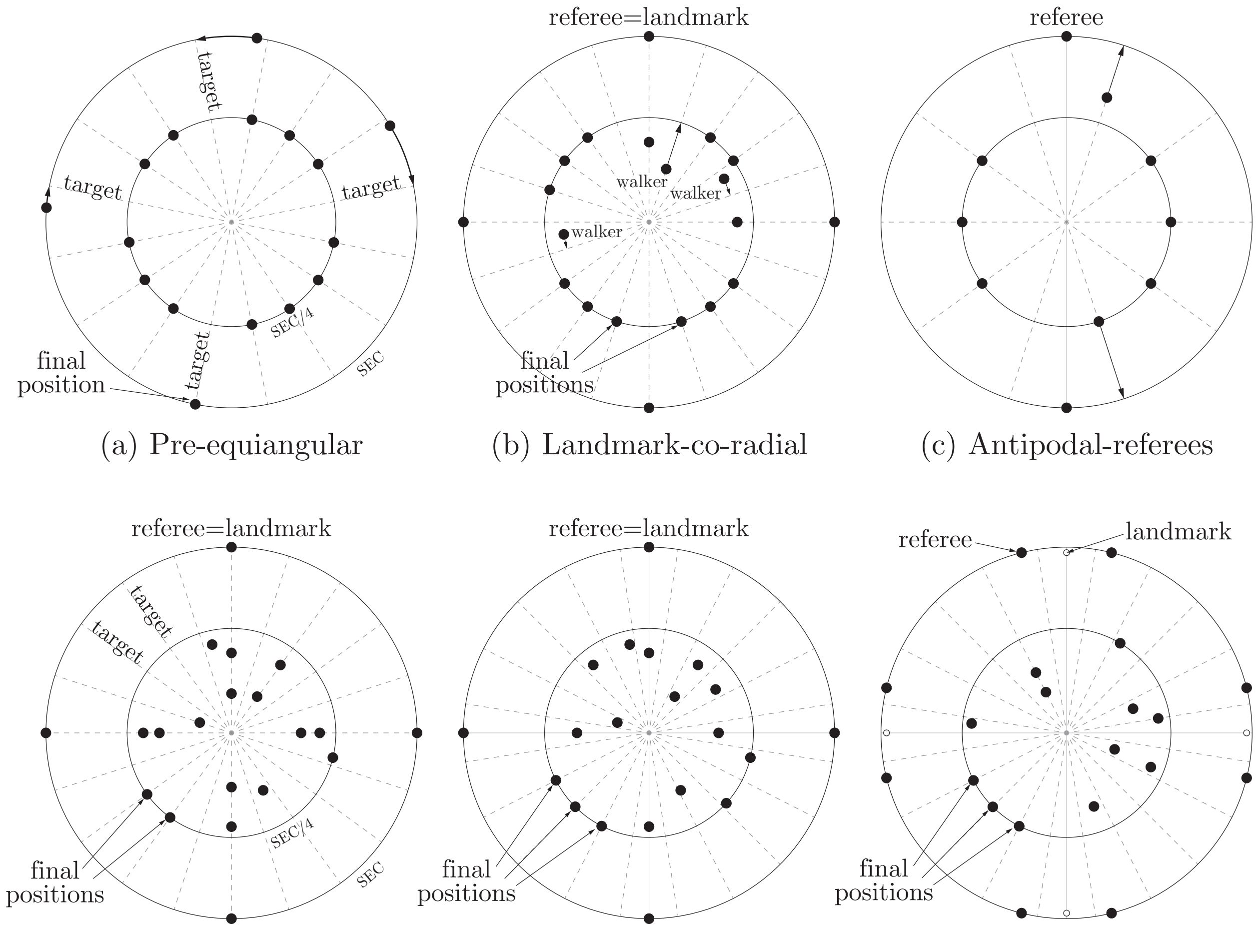
3.4 The Intermediate Tests

Having failed the initial tests, the next sequence of tests is for the classes of configurations defined below, which can occur as the initial configuration, or as an evolution from a *Periodic* configuration. Along with the definitions, the actions to perform in each configuration are given.

Pre-equiangular. There are robots both on SEC and on SEC/4, and nowhere else. The robots on SEC are at least three, and those on SEC/4 are forming an "almost" *Regular* configuration; that is, a *Regular* with one missing point for each robot on SEC. The missing points may be arranged in two different ways. They may form a "regular pairs" arrangement, in which there are pairs of missing points in adjacent positions, in such a way that the pairs are equally spaced around SEC/4; otherwise, they form a "regular pairs" arrangement in which exactly one element of each pair has been removed. There is a bijection between robots on SEC and missing points, determined by the minimum total distance the robots on SEC must travel to occupy them (Figure 2(a) shows an arrangement on SEC/4 of the second type).

In this case, the robots on SEC rotate towards their targets, which are uniquely determined by the positions of the robots on SEC/4. With this action, called procedure PRE-EQUIANGULAR, the robots eventually reach an *Equiangular* configuration.

Landmark-co-radial. The robots on SEC form an *Equiangular* set, and these are the referees, which also coincide with the landmarks. The landmarks define the n target half-lines, in such a way that either all landmarks lie on some targets (as in Figure 2(b)), or they lie on bisectors of adjacent targets. All the non-referee robots are on or inside SEC/4: each robot on SEC/4 is on a target; the only ones *strictly* inside are those co-radial with the referees,



(a) Pre-equiangular (b) Landmark-co-radial (c) Antipodal-referees

(d) Post-periodic (e) Post-periodic (f) Post-periodic

Figure 2: Examples of the possible evolutions of a *Periodic* configuration

and at most one robot (called *walker*) for each referee. The central targets of each sector defined by two adjacent referees are all occupied by robots on SEC/4 in such a way that, for each landmark, the open sector Γ defined by the nearest target in the clockwise direction that is occupied by a robot on SEC/4 and the nearest one in the counterclockwise direction contains as many robots as targets. Moreover, Γ contains at most one walker, and the targets in Γ that lie to the left of the landmark differ by at most one unit from those to the right. A *co-radial Biangular* configuration falls in this class, too.

In this configuration, the intended behavior is to “resolve” all the robots that are co-radial to the referees, and have them move to their targets, reaching an *Equiangular* or a *Pre-equiangular* configuration (depending if the referees are already on their targets or not).

Note that in a *Landmark-co-radial* the only unoccupied targets correspond to the groups of co-radial robots of the landmarks and to at most one robot per landmark, the *walker*, which is moving towards a target. The co-radial robots move in turns. If there is no walker in the sector Γ (as defined above) around a landmark, the most internal non-referee that is co-radial with that landmark rotates toward the farthest away target among those in Γ, becoming a *walker*. When a walker reaches its target, it moves radially to reach SEC/4. If all the non-referees are on their targets, they all lie on SEC/4, and the configuration happens to be *Antipodal-referees* (see below), then the two non-referees closest to the landmarks move toward SEC (thus "forcing" the configuration to transition into an *Antipodal-referees* configuration that is not a *Landmark-co-radial* anymore, which is tested after *Landmark-co-radial* by the algorithm). Otherwise, the configuration becomes either *Equiangular* or *Pre-equiangular*, as intended.

Post-periodic. The robots on SEC form an *Equiangular* or a *Simple biangular* set, and they are the referees. All other robots lie on SEC/4 or inside of it. If the referees are *Equiangular*, the landmarks coincide with the referees and they all have the same number of co-radial robots, which lie *strictly* inside SEC/4. If the referees are *Biangular*, the landmarks are the midpoints of neighboring referees, and no robot is co-radial with any landmark. The robots that are not co-radial with the landmarks are equidistributed among the sectors defined by the landmarks.

In this configuration, the targets are calculated with respect to the landmarks, depending on the parity of the robots that are co-radial with each landmark (including the referees): if they are odd, then the landmarks lie on some targets (Figure 2(d)); if they are even (which includes zero), the landmarks lie on bisectors of adjacent targets (Figures 2(e) and 2(f)). Note that, if such co-radials are odd, the referees must be *Equiangular*. Each robot may be associated with a unique target, or to two possible targets (in case of left-right symmetry of its view).

The intended behavior in a *Post-periodic* configuration is to have all robots move onto SEC/4 on their respective targets, except for the robots that are co-radial with some landmark, thus reaching a *Landmark-co-radial* configuration. To do so, the non-referees that are not co-radial with the landmarks and that can reach SEC/4 without colliding with other robots, move radially toward it. If none can do it and there are co-radial robots that are not co-radial with any landmark, the most internal of these co-radials rotates in an arbitrary direction of 1/4 of the minimum non-zero element in μ. If all the non-referees that are not co-radial with the landmarks are already on SEC/4, they orderly rotate on SEC/4 until they reach their targets (which are now uniquely determined). This is done in such a way that only the robots that can reach their target without colliding with other robots move. Each move is cautious, with critical points corresponding to *Landmark-co-radial* and *Pre-equiangular* configurations. At this point, the configuration becomes: *Landmark-co-radial* if the referees are *Equiangular* and have co-radial robots; *Pre-equiangular* if the referees are *Biangular* and they are not on their targets; or *Equiangular* if the referees are *Equiangular* and there are not co-radial robots, or if they are *Biangular* and already on their targets.

Antipodal-referees. There are two antipodal robots on SEC, which are the referees. On SEC/4 there are (possibly among others) $n-4$ robots that are forming a *Regular* configuration with some missing points. More precisely there are two antipodal pairs of adjacent missing points, such that each referee is equidistant to two adjacent missing points. Furthermore, there are two other robots co-radial with two non-adjacent missing points, which lie between SEC/4 and SEC (possibly on SEC/4 or on SEC). Note that this configuration is uniquely identifiable and has period either n or $n/2$ (see Figure 2(c)). In this configuration, the robots closest to the referees (one for each referee) move towards SEC, eventually reaching a *Pre-equiangular* configuration.

Simple biangular. In this case, the intended behavior of the robots is to reach a *Pre-regular* configuration by moving toward SEC according to the cautious move protocol, with critical points on SEC/4 (where a *Landmark-co-radial* or a *Pre-equiangular* may be formed), and additional critical points where *Pre-regular* configurations may be formed (see Theorem 5). If the robots already on SEC belong to the same analogy class, the other robots in the same class move first.

3.5 The Periodic Test

Periodic. If the procedure PERIODIC is executed, it means that the configuration is *Periodic*, and additionally it does not belong to any of the classes described above. In this case, the intended behavior is to elect the referees, define the landmarks, have the referees move onto SEC and the non-referees move into SEC/4, reaching a *Post-periodic* configuration. In trying to do this, the robots can find themselves in a variety of different configurations, and the algorithm might switch to several different cases.

Let k be the period. If there exist robots with exactly n/k homologous robots, then the lex-first among these robots are chosen to be the referees, as well as the landmarks. If this is not the case, all homology classes must have size exactly $2n/k$. If the robots in $\mathcal{L}_1$ are not *Equiangular* (and therefore they are strictly *Biangular*), they are chosen to be the referees; otherwise the referees are the robots in $\mathcal{L}_2$. Note that in both cases the referees form a *Simple biangular* set; the landmarks are selected to be the midpoints of neighboring referees. Hence, by construction, the landmarks are always n/k points forming an *Equiangular* set (with respect to the center of the SEC of all robots), and they define n/k sectors, each containing the same number of robots in its interior.

If the configuration is *Double-biangular*, no referee is on SEC, and some non-referees are not on SEC, then all the non-referees move radially to reach SEC. Otherwise, if there are referees not on SEC, they move radially to reach SEC. If all the referees are on SEC, the other robots move radially inward until they reach SEC/4 or its interior. All non-referee robots that are co-radial with some landmark move *strictly* inside SEC/4. The non-referee robots move in turns, in such a way that only homologous robots move simultaneously. Specifically, the non-referees that belong to homology classes of size n/k move first.

In all cases, all movements are cautious, with critical points on SEC/4 (which may yield a transition into *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular*), and those determined by *Pre-regular* configurations (see Theorem 6).

When this is done, the configuration becomes *Post-periodic*, with some exceptions: if the robots not co-radial with the referees are already on their targets on SEC/4, except at most one per landmark, the configuration becomes *Landmark-co-radial*; if the only robots not on their targets are the referees, and the referees are more than two, the configuration becomes *Pre-equiangular*; if the only robots not on their targets are the referees, and the referees are only two, the configuration becomes *Antipodal-referees*.

3.6 The Aperiodic Tests

In this last set of tests, *Post-aperiodic* and *Aperiodic* configurations are addressed. Similarly to the previous cases, the intended behavior of the actions is to elect the referees and to identify landmarks and targets. From the *Aperiodic* configuration, the intended behavior is to reach a *Post-aperiodic* configuration and, from there, an *Equiangular* configuration.

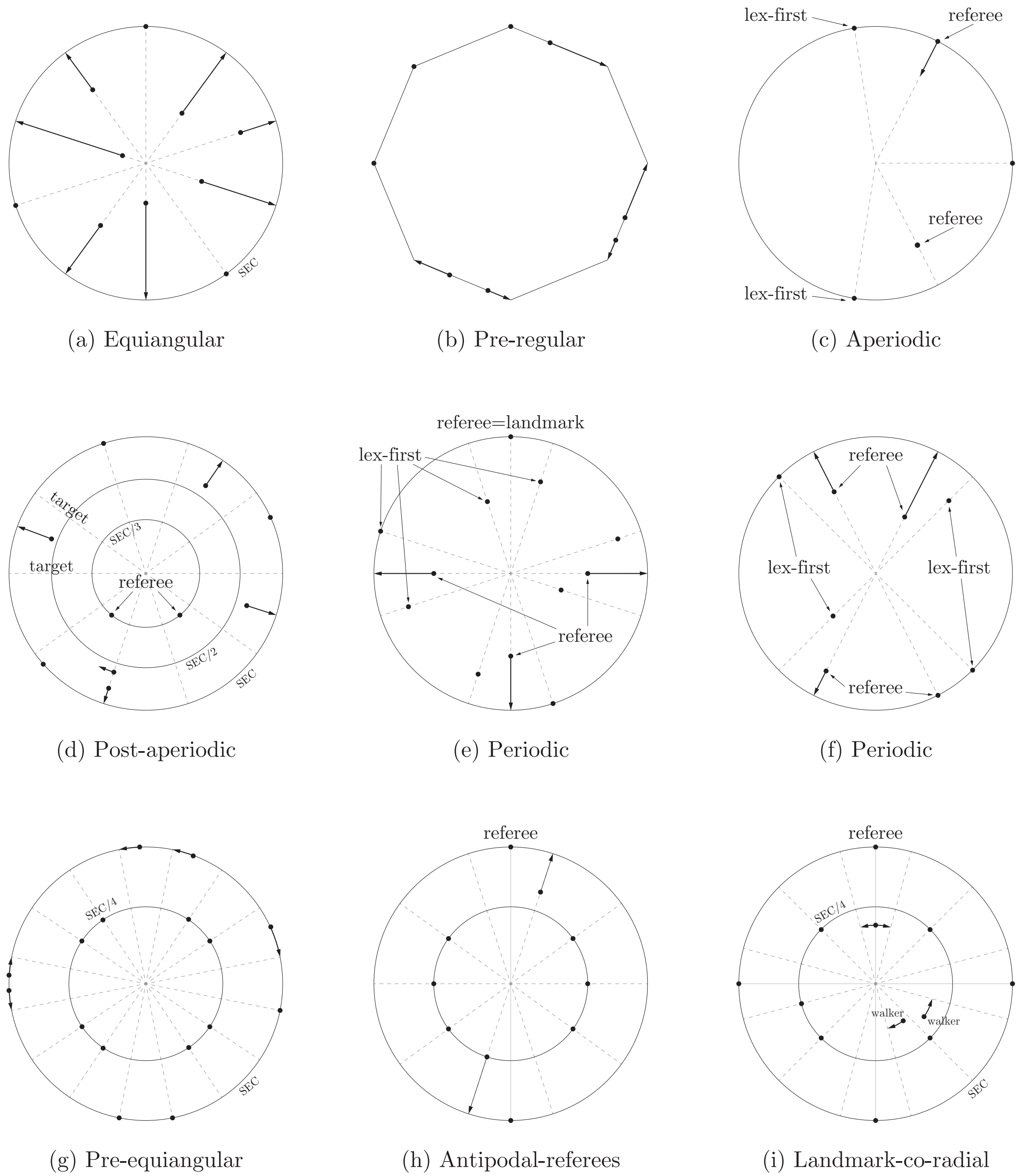
Post-aperiodic. There are either one or two robots on SEC/3, which are the referees. All other robots are found between SEC/2 and SEC. If there are two referees, they are not antipodal (i.e., their midpoint is not the center of SEC).

In this configuration, the actions taken by the robots (procedure POST-APERIODIC) are as follows. If there are two referees, and all the non-referees are on SEC forming a *Regular* set with two adjacent missing points, the two referees rotate on SEC/3 until they become co-radial with the missing points, and the configuration becomes *Equiangular*. Otherwise, the targets are identified by the referees on SEC/3, and a unique target is assigned to each robot. The non-referees that can move radially to SEC without colliding, do so. If there are non-referees that cannot radially move to SEC (because other robots are in their way), then the most internal non-referees rotate of 1/4 of the minimum non-zero element of μ to remove the co-radiality. If all the non-referees are on SEC and there is only one referee, the non-referees cautiously rotate to their respective targets, in such a way that SEC never changes and no two robots collide, and using *Simple biangular* and *Periodic* configurations as critical points. If the targets are reached, the configuration becomes *Equiangular*. Finally, if there are two referees, and all non-referees are on SEC, not forming a *Regular* set with two adjacent missing points, the non-referees rotate on SEC with a cautious move as in the previous case, with additional critical points given by the configurations in which the robots on SEC form a *Regular* set with two adjacent missing points. In this last case, the configuration may become *Simple biangular*, *Periodic*, *Equiangular*, or remain *Post-aperiodic*.

Aperiodic. The procedure APERIODIC is executed if the current configuration fails all previous tests. If the configuration is *co-radial uni-aperiodic*, then the lex-first is unique, and must have co-radial robots. In this case the referee is the most internal among the robots that are co-radial with the lex-first. If the configuration is *non-co-radial uni-aperiodic*, the lex-first is still unique, but it may be necessary to keep SEC intact. If this is not the case, the lex-first is the referee, otherwise the referee is the lex-second (it is easy to see that, if $n \geqslant 5$, one of these two robots can be removed without altering SEC).

If the configuration is *co-radial bi-aperiodic*, let r and r' be, among the robots that are co-radial with the lex-first robots, the most internal ones, respectively. If r and r' are not aligned with the center of SEC, then they are chosen to be the referees. Otherwise, the referees are the first two robots in the lexicographically minimum order (which are homologous) that can be *safely* removed without altering SEC (assuming that all robots that can reach SEC radially are already on SEC), and such that they are the most internal robots among their co-radials. (Note that, in some configurations, these referees happen to be the same robot. In these cases, the referee is unique.)

Finally, if the configuration is *non-co-radial bi-aperiodic*, the referees are the first two (just one, in some special cases) homologous robots that are not aligned with the center of



(a) Equiangular (b) Pre-regular (c) Aperiodic

(d) Post-aperiodic (e) Periodic (f) Periodic

(g) Pre-equiangular (h) Antipodal-referees (i) Landmark-co-radial

Figure 3: Some examples of configurations

SEC, and such that, when all robots are on SEC, they can be removed without changing SEC.

The non-referees that are inside or on SEC/2 move out of SEC/2. Those that can reach SEC without colliding, do so. They take turns in such a way that only homologous robots move simultaneously (hence at most two), and they move radially outward, performing a cautious move with critical points on SEC/4, SEC/3, SEC/2, SEC, and those determined by the *Pre-regular* configurations (see Theorem 7). During these movements, the configuration may become either *Post-periodic*, *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular* (when they pass through SEC/4 or when they reach SEC), or *Post-aperiodic* (when they pass through SEC/3), or *Pre-regular*. Otherwise, the configuration stays *Aperiodic*. If all the non-referees are outside SEC/2 and none of them can move to SEC without colliding, the referees move and reach SEC/3. They use a cautious move with SEC/4 as a critical point, and those determined by the *Pre-regular* configurations (see Theorem 7). The configuration may become *Post-periodic*, *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular* (when the robots reach SEC/4), or *Pre-regular*. Otherwise it becomes *Post-aperiodic*, as intended.

4 Properties and Correctness

To prove the correctness of the algorithm, we need to analyze the possible transitions between configurations. Some transitions come as a result of the "intended" behavior of the robots executing the algorithm; other transitions come as "incidental" byproducts of the execution, and have to be analyzed as well.

In the following, we will closely examine all the possible flows of the algorithm in the space of robots' configurations, paying special attention to the transitions that may arise as critical points of cautious moves. In Section 4.1 we prove some fundamental results on cautious moves, which show that robots executing the cautious move protocol introduced in Section 3.2 indeed behave as intended. In Section 4.2 we thoroughly analyze the *Pre-regular* case, which turns out to be the hardest to treat. Then, in Section 4.3 we conclude the proof by showing that all the possible flows of the algorithm eventually reach a *Regular* configuration.

In this section, unless stated otherwise, $\mathcal{R} = \{r_1, \cdots, r_n\}$ will denote a swarm of $n > 4$ robots. By $r_i(t)$ we denote the location of robot r_i at time $t \geqslant 0$, and we let $\mathcal{R}(t) = \{r_1(t), \cdots, r_n(t)\}$.

4.1 Correctness of Cautious Moves

Let a set of robots execute the cautious move protocol of Section 3.2, starting from a given configuration I and using a set of critical points C. We denote by $\mathcal{E}^{\delta}_{I,C}$ the set of all possible *executions* of such a robot system, where δ is the minimality constant introduced in Section 1. An execution is just the sequence of configurations reached by the robots as a function of time, which depends on how the adversary activates the robots. Note that, if $0 < \delta' \leqslant \delta$, then $\mathcal{E}^{\delta}_{I,C} \subseteq \mathcal{E}^{\delta'}_{I,C}$. Since δ is not known to the robots, and an algorithm solving a problem must do so for every choice of $\delta > 0$, it makes sense to consider the set $\mathcal{E}_{I,C} = \bigcup_{\delta>0} \mathcal{E}^{\delta}_{I,C}$ as the class of all possible executions, regardless of how small the constant δ is. Let a *property* of executions be any set of executions. We say that a cautious move with critical point set C and initial configuration I *enjoys* the property $\mathcal{P}$ if $\mathcal{E}_{I,C} \subseteq \mathcal{P}$.

First we show that a cautious move always "terminates", that is, if every robot has a path of finite length containing finitely many critical points, then after finitely many turns it reaches the end.

Lemma 1. *Let a swarm of robots execute the cautious move protocol from an initial configuration in which no robot is moving. Then, in a finite amount of time each robot will be found at the endpoint of its path, and no robot will be moving.*

Proof. Let a *round* be a span of time in which every robot executes at least one complete cycle. Any execution can be decomposed into an infinite sequence of rounds. Let $L(t)$ be the set of robots that are farthest from the endpoints of their respective paths at time t, and let $d(t)$ be the distance of any robot in $L(t)$ from the endpoint of its path at time t. Suppose by contradiction that $d(t) > 0$ for every t. Since $d(t)$ can only decrease in time, it converges to an infimum m. Suppose first that the infimum is reached, i.e., $d(t) = m$ for some t. Then, after a round, say at time t', all the robots in $L(t)$ have moved, and hence $d(t') < m$, which is a contradiction.

Suppose that $d(t) > m$ for every t, and therefore the infimum is never reached. Let t' be such that $d(t') - m < \delta$. Let $r \in L(t')$ and let $r(\infty)$ be the point on r's path at distance m from the endpoint. Since the critical points are finitely many, we may assume that no critical points (auxiliary or not) and no midpoints of consecutive critical points lie on the path of r strictly between $r(t')$ and $r(\infty)$. By our choice of t', all the robots that perform a cycle at any time after t' necessarily reach their destination point. At every time t, let $d_i(t)$ be the distance of the i-th robot from its endpoint. Let $c(t)$ be the number of different values greater than m that $d_i(t)$ takes as $1 \leqslant i \leqslant n$. By the above observations and by the protocol, after every round $c(t)$ decreases by at least 1, until finally $c(t'') = 1$. At this point, whenever a robot in $L(t'')$ moves, it reaches a distance from the endpoint of its path that is at most m. Hence, after a round, at time t''', $d(t''') \leqslant m$, which is a contradiction.

It follows that each robot eventually reaches the endpoint of its path. Since this is also a critical point and the robot is not moving in the initial configuration, it stops there. Afterwards, every time the robot performs a Look and some other robot has not reached the endpoint of its path yet, it waits. Eventually, when the last robots have reached the endpoints of their paths and they stop, none of the robots is moving. □

Next we prove that cautious moves are sound, i.e., that if a configuration of points C is taken as the set of critical points of a cautious move, then, whenever the robots are found in configuration C, they simultaneously stop.

Theorem 1. *Let a swarm of n robots execute a cautious move with critical points C, with $|C| = n$, from an initial configuration in which no robot is moving. Then, during the cautious move, whenever the robots are found in configuration C, they simultaneously stop.*

Proof. Because the paths of the n robots are disjoint, C can be attained only if each path contains exactly one point of C. By c_r we denote the element of C that lies on the path of robot r. Since a robot can only move toward the endpoint of its path, we may assume that each robot r is initially located not past c_r along its path, otherwise C would never be attained during the cautious move. Let d_r be the distance between c_r and the endpoint of r's path (measured along r's path), and let F be the set of robots r such that d_r is minimum.

Suppose first that each robot $r \notin F$ initially lies at c_r. According to the cautious move protocol, the only robots that are able to move in this situation are those in F. By Lemma 1, for every $r \in F$ there exists a minimum time t_r such that $r(t_r) = c_r$. Since this is a critical point, r stops in c_r at time t_r. Moreover, r waits in c_r until time $t^* = \max_{r \in F}\{t_r\}$. Therefore, at time t^*, the robots form configuration C for the first time, and none of them is moving. After that time, as soon as a robot r moves, it passes c_r, and therefore C cannot be formed any more.

Suppose now that some robots *not* in F initially lie strictly before the element of C on their respective path. For every $r \notin F$, let f_r be the "auxiliary" critical point on the path of r that corresponds to $c_{r'}$, with $r' \in F$. Let F' be the set of robots $r \notin F$ such that r is initially located in f_r or before f_r. Let F'' be the set of robots that are not in F or in F'. By our assumptions, $F' \cup F''$ is not empty. For every $r \in F$, we define t_r as in the previous paragraph. For $r' \in F'$, we define $t_{r'}$ as the minimum time at which r' is located at $f_{r'}$. Finally, we let $t^* = \max_{r \in F \cup F'}\{t_r\}$. By the cautious move protocol, for every $r \in F$, $r(t^*) = c_r$ and, for every $r' \in F'$, $r'(t^*) = f_{r'}$. On the other hand, until time t^*, no robot in F'' has moved.

If F' is empty, then some robot $r \in F''$ is not located in c_r at time t^* or before time t^*. Hence, the robots cannot form configuration C until time t^*. After time t^*, the first robots that move are those in F. When one of these robots moves, it goes past the element of C that lies on its path, and therefore C cannot be formed after time t^*, either.

Let F' be not empty. Then, the robots cannot form configuration C until time t^*, because each robot $r \in F'$ is located strictly before c_r at all times $t \leqslant t^*$. After time t^*, the first robots that are allowed to move are those in $F \cup F'$. For each $r \in F \cup F'$, let $t'_r \geqslant t^*$ be the first time at which robot r performs a Look. Since $r(t')$ coincides with a (possibly auxiliary) critical point, by the cautious move algorithm the destination point of r is strictly before the next critical point of r, and strictly after its current position. In particular, if $r \in F'$, its destination point is strictly before c_r. After such a robot r has moved, it waits at least until after time $\max_{r \in F \cup F'}\{t'_r\}$. Indeed, before r moves again, all the robots in $F \cup F'$ must "catch up" with r. However, as soon as a robot $r' \in F$ moves after time t^*, it goes past $c_{r'}$, and therefore the configuration C is not formable any more. □

Now we show that the cautious move protocol is "robust", in that merging two sets of critical points yields a cautious move that enjoys all the properties that are enjoyed when either set of critical points is taken individually.

Lemma 2. *$\mathcal{E}_{I,C\cup\{p\}} \subseteq \mathcal{E}_{I,C}$.*

Proof. By the cautious move protocol, the addition of p to the path of one robot causes the appearance of at most one extra critical point on the path of each other robot. However, by Lemma 1, each robot still reaches the end of its path within finitely many turns in every execution. Let $E \in \mathcal{E}_{I,C\cup\{p\}}$ be an execution. We claim that $E \in \mathcal{E}^{\delta}_{I,C}$, for a suitable choice of a small-enough δ. Let us order chronologically the Look phases of all the robots in the execution E (we may assume Look phases to be instantaneous), and let us prove by induction that, up to the k-th Look, E coincides with some execution in $\mathcal{E}_{I,C}$.

The claim is certainly true for $k = 1$. Let us assume it is true for $k \geqslant 1$ and let us prove that it holds for $k + 1$. Let r be the robot performing the k-th Look at time t, and let it compute d as a destination point according to the cautious move protocol. If d is originating from C (i.e., it is either a "corresponding" point of some point in C, or the midpoint between two consecutive such points), then our claim holds. Similarly, if d is the location on r's path "corresponding" to some other robot, our claim holds. If none of the above holds, d may be either p, or the "corresponding" point of p on r's path, or the midpoint between p and another critical point of C (or between their corresponding points on r's path).

Let d' be the destination point computed by r in the same configuration, but with critical point set C instead of $C \cup \{p\}$. If d' is not closer to $r(t)$ than d (along r's path), we let δ be the distance between $r(t)$ and d. Clearly, with this choice of δ, the adversary may decide to stop r's next Move phase as soon as r has reached d, no matter if r has computed a farther destination point. By inductive hypothesis, E coincides with some execution in $\mathcal{E}_{I,C}$, and therefore with some execution in $\mathcal{E}^{\delta'}_{I,C}$, for some $\delta' > 0$. Hence, in this case E coincides with some execution in $\mathcal{E}^{\min\{\delta,\delta'\}}_{I,C}$, and therefore in $\mathcal{E}_{I,C}$, up to the $(k+1)$-th Look phase.

Finally, suppose that d' is closer than d to $r(t)$. We argue that this case cannot occur. Let a be the last (possibly auxiliary) critical point corresponding to a point of C that r has encountered on its path. If such a critical point does not exist because r has not encountered critical points yet, then we let a be the starting point of r's path. Furthermore, let b be the next critical point corresponding to a point of C. It is easy to see that $r(t)$, d', and d (in this order) belong to the sub-path of r going from a to b. Also, since $d \neq d'$, there must be a correspondent of p, say p', strictly between a and b. Let m be the midpoint of ab, let m_1 be the midpoint of ap', and let m_2 be the midpoint of $p'b$. Clearly, m lies strictly between m_1 and m_2. If $r(t) = a$ and a is a critical point, then $d = m_1$ and $d' = m$, hence d' cannot be closer to $r(t)$ than d. Otherwise, $d' = b$, and hence d cannot be farther than d' from $r(t)$. □

Theorem 2. *Let the cautious move with initial configuration I and critical point set C_1 (resp. C_2) enjoy property $\mathcal{P}_1$ (resp. $\mathcal{P}_2$). Then, the cautious move with initial configuration I and critical point set $C_1 \cup C_2$ enjoys both $\mathcal{P}_1$ and $\mathcal{P}_2$.*

Proof. The theorem easily follows from Lemma 2: we add the critical points of C_2 to the set C_1, one by one. Each time we add a new point, by Lemma 2 we have a set of executions that is a subset of the previous one, and therefore is still a subset of $\mathcal{P}_1$. Hence the cautious move with critical points $C_1 \cup C_2$ enjoys property $\mathcal{P}_1$ and, by a symmetric argument, it also enjoys property $\mathcal{P}_2$. □

Corollary 1. *Let a swarm of n robots execute a cautious move with critical point set $\bigcup_{i=1}^{k} C_i$, with $|C_i| = n$ for $1 \leqslant i \leqslant k$, from an initial configuration in which no robot is moving. Then, during the cautious move, whenever the robots are found in a configuration C_i, they simultaneously stop.*

Proof. By Theorem 1, the cautious move with critical point set C_i has the property $\mathcal{P}_i$ that, whenever the robots are found in configuration C_i, they simultaneously stop. By repeatedly applying Theorem 2, we have that the cautious move with critical point set $\bigcup_{i=1}^{k} C_i$ enjoys all properties $\mathcal{P}_i$, for every i. □

4.2 Analysis of Pre-Regular Configurations

In this section, we prove several properties of *Pre-regular* configurations that will be needed in the correctness proof of Section 4.3. First we show that in a *Pre-regular* configuration no two points are co-radial (Corollary 2) and that no point lies in SEC/3 (Theorem 4). Then we prove that *Pre-regular* configurations can effectively be taken as critical points during the execution of the algorithm, by showing that only finitely many *Pre-regular* configurations are formable whenever a cautious move has to be made, or that the "relevant" *Pre-regular* configurations that are formable are only finitely many.

From our observations it will also follow that it is easy for the robots to test if a given configuration of $n > 4$ points is *Pre-regular*. Indeed, if n is odd or the points are not in a strictly convex position, then they do not form a *Pre-regular* configuration, by Observations 2. Otherwise, the pairs of companions can be identified thanks to Observation 4. Since the set of companion pairs determines the supporting polygon of a *Pre-regular* configuration, it is now easy to compute such a polygon and verify if it is indeed regular.

In the following, we assume that $S \subset \mathbb{R}^2$ is a finite set of $n > 4$ points, none of which lies at the center of SEC. In particular, if S is *Pre-regular*, then $n \geqslant 6$, because in this case n must be even. Since points model robots' locations, with abuse of terminology we will refer to points of S that "slide" according to some rules. Formally, what we mean is that we consider S as a function of time, so that $S(t)$ represents a set of robots' locations at time t; likewise a "sliding" point $a \in S$ will formally be a function $a(t)$ representing the trajectory of a robot.

4.2.1 Co-Radial Points and Points Inside SEC/3

Observation 1. *The center of the SEC of S lies in the convex hull of the points of S that lie on SEC. Therefore, every half-circle of SEC contains some points of S. In particular, if just two points of S lie on SEC, they are antipodal.*

Observation 2. *If S is* Pre-regular*, then S is in strictly convex position, and in particular no three points of S are collinear. Moreover, the convex hull of S contains the center of the supporting polygon of S.*

Theorem 3. *If S is* Pre-regular*, then any ray from the center of SEC intersects the perimeter of the supporting polygon in exactly one point.*

Proof. Let a ray from the center of SEC intersect the perimeter of the supporting polygon in exactly two points a and b, none of which coincides with the center of SEC. Then, by Observation 2, the intersection of the line through a and b with the convex hull of S is exactly the segment ab, and therefore the center of SEC does not belong to the convex hull of S. This contradicts Observation 1.

Suppose now that an edge of the supporting polygon, belonging to a line ℓ, is collinear with the center of the SEC of S. Due to Observation 2, S lies entirely on one side of ℓ (or on ℓ itself), and therefore in one half of SEC. Because of Observation 1, there must be two antipodal points of S lying at the intersections between ℓ and the perimeter of SEC. Considering that the supporting polygon is a regular polygon, this implies that it has no other intersections with SEC (as its edges are at least as long as the diameter of SEC), which means that $n = 2$, contradicting our assumption then $n > 4$. $\square$

Corollary 2. *If S is* Pre-regular*, then no two points of S are co-radial (with respect to the center of SEC).*

Proof. If two points of $a, b \in S$ were co-radial, then the ray from the center of SEC through a and b would intersect the perimeter of the supporting polygon in at least a and b, contradicting Theorem 3. $\square$

Lemma 3. *There exist two points of S at angular distance at least $2\pi/3$ lying on SEC.*

Proof. Let $a \in S$ be a point on SEC, and let b and c be the two points on SEC at angular distance $2\pi/3$ from a. Further, let d and e be the two points on SEC at angular distance $\pi/2$ from a, such that d is closer to b and e is closer to c. If some point of S lies on the arc bc, it has distance at least $2\pi/3$ from a, and the lemma follows. Otherwise, one of the arcs bd and ce must contain a point of S, by Observation 1. Without loss of generality, let $f \in S$ belong to the arc bd, and let f' be the point of SEC antipodal to f. Again by Observation 1, there must be a point $g \in S$ on the arc cf'. Then, f and g have angular distance at least $2\pi/3$. $\square$

Observation 3. *Let a circle with radius r and an annulus with inner radius strictly greater than r be given in the plane. Then, at least a half-annulus lies outside the circle.*

Theorem 4. *If S is* Pre-regular*, then no points of S lie on or inside SEC/3.*

Proof. By contradiction, let S be *Pre-regular*, and let $a \in S$ be a point on or inside SEC/3. Let $b, c \in S$ be two points on SEC at angular distance at least $2\pi/3$, which exist by Lemma 3. The angle $\angle bac$ is minimized when the angular distance between b and c is exactly $2\pi/3$, and a lies on the axis of bc, on the farthest point from b and c. In this case $\angle bac > 0.51\pi$, and therefore abc is always an obtuse triangle. Also, if the radius of SEC is r, then the length of both ab and ac is at least $2r/3$.

Because S is *Pre-regular*, all points of S lie on the same regular n-gon, with $n \geqslant 6$. Therefore, they all lie in an annulus with inner and outer radiuses r' and r'' respectively, such that $r''/r' \leqslant 2/\sqrt{3}$. In the following, we will prove that $r' > r$. This will imply, by Observation 3, that at least half of the annulus lies outside SEC, and therefore it is devoid of points of S. This is a contradiction, because every other edge of the supporting polygon must contain points of S and, because $n \geqslant 6$, every half-annulus contains at least two whole adjacent edges of the supporting polygon.

Let $\mathcal{A}$ be the set of all annuli that contain a, b, c, whose inner radius is at most r, and whose ratio between outer radius and inner radius is at most $2/\sqrt{3}$. With the standard topology, $\mathcal{A}$ is a compact set, and the function $\rho : \mathcal{A} \to \mathbb{R}$ mapping an annulus of $\mathcal{A}$ into the length of its inner radius is continuous. Therefore, by Weierstrass' extreme value theorem, there exists an annulus in $\mathcal{A}$ with minimum inner radius. Given an annulus $A \in \mathcal{A}$, we will prove that either $\rho(A)$ is not minimum, or $\rho(A) > r$. This will imply that $\mathcal{A}$ is empty and, by the previous paragraph's reasoning, it will conclude the proof.

If at least two among a, b, and c lie strictly inside A (i.e., in the region strictly between the inner and the outer circles), then we can shrink A about the third point while keeping all three points in the annulus. This yields another annulus of $\mathcal{A}$, with strictly smaller inner radius. Hence $\rho(A)$ is not minimum.

If exactly one among a, b, c, say a, lies strictly inside A, we can slightly rotate A about b in the proper direction, in such a way that both a and c end up strictly inside the annulus. The new annulus is still in $\mathcal{A}$, has the same inner radius as A and, by the previous paragraph, its inner radius is not minimum.

Let all of a, b, c lie on the boundary of A. If they all lie on the inner circle or on the outer circle, then they also lie on the same half-circle, because abc is an obtuse triangle. Hence it is possible to slightly translate A in such a way that all of a, b, c get strictly inside the annulus. By the above paragraphs, $\rho(A)$ is not minimum.

Otherwise, without loss of generality, a and b lie on different components of the boundary of A (i.e., one lies on the inner circle, the other on the outer circle). The length of ab is maximum when the line ab is tangent to the inner circle. Since the outer radius of A is at most $2\rho(A)/\sqrt{3}$, by Pythagoras' theorem it follows that $ab \leqslant \rho(A)/\sqrt{3}$. But recall that $ab \geqslant 2r/3$, which yields $\rho(A) \geqslant 2r/\sqrt{3} > r$. □

Corollary 3. *If S is* Pre-regular*, then it is not* Central*,* Post-periodic*,* Landmark-co-radial*,* Antipodal-referees*,* Pre-equiangular*, nor* Post-aperiodic*.*

Proof. All these configurations contain points on or inside SEC/3, hence they are not *Pre-regular* due to Theorem 4. □

4.2.2 Cautious Moves for Simple Biangular Configurations

Observation 4. *If S is* Pre-regular *and x, y are companions, then $xz \geqslant xy$ for every $z \in S \setminus \{x\}$. Moreover, if c is the center of the supporting polygon, then $\angle xcz \geqslant \angle xcy$ for every $z \in S \setminus \{x\}$.*

Lemma 4. *If S is* Pre-regular*, the cyclic order of S around the center of SEC is the same as the cyclic order of S around the center of the supporting polygon.*

Proof. By Observation 2, S is in convex position, hence any two points in the convex hull of S induce the same cyclic order on S. By Observation 1, the center of SEC lies in the convex hull of a subset of S, hence it lies in the convex hull of S. But due to Observation 2, the center of the supporting polygon is contained in the convex hull of S as well, hence the claim follows. □

Lemma 5. *If some points of S are allowed to "slide" radially in such a way that SEC never changes and there are at least three consecutive points $a, b, c \in S$ (in this order) that do not slide, with $ab = bc$, then there is at most one configuration of the points that could be* Pre-regular*.*

Proof. If some configuration is *Pre-regular*, then by Lemma 4 either a and b are companions, or b and c are. Since $ab = bc$, then ab and bc are adjacent edges of the supporting polygon, and therefore the whole supporting polygon is fixed, no matter how the points slide. Then, there is only one possible position in which each sliding point may lie on the supporting polygon, due to Theorem 3. Hence, if a *Pre-regular* configuration is formable, it is unique. □

Observation 5. *For every $n \geqslant 3$, if three straight lines are given in the plane, there is at most one regular n-gon with three edges lying on the three lines.*

Lemma 6. *If some points of S are allowed to "slide" radially in such a way that SEC never changes, and there are at least three consecutive points $a, b, c \in S$ (in this order) that do not slide, plus at least another non-sliding point d, not adjacent to a nor c, then there is at most one configuration of the points that could be* Pre-regular*.*

Proof. If some configuration is *Pre-regular*, then by Lemma 4 either a and b are companions, or b and c are. If $ab = bc$, Lemma 5 applies. Otherwise, without loss of generality, assume that $ab < bc$, and therefore a and b are companions, due to Observation 4. Then all the companionships are fixed, again by Lemma 4. The slope of the edge of the supporting polygon through a and b is fixed, hence all the slopes of the other edges are fixed, because the supporting polygon is regular. In particular, the slopes of the edges through c and d are fixed, and these are two distinct edges because c and d are not adjacent. Therefore, by Observation 5, the whole supporting polygon is fixed. It follows that there is at most one position of the sliding points that could be *Pre-regular*, due to Theorem 3. □

Lemma 7. *If S* Pre-regular*, then every internal angle of the convex hull of S is greater than $\pi(n-3)/n$.*

Proof. Let x, y, z, w be four consecutive vertices of the convex hull of S, such that x is the companion of y and z is the companion of w. Let ab be the edge of the supporting polygon containing x and y, such that x is closer to a. Similarly, cd is the edge containing z and w, and z is closer to c. The infimum of $\angle xyz$ is reached when y coincides with b, w coincides with d, and z tends to w. As the limit angle contains exactly $n-3$ edges of the supporting polygon, its size is $\pi(n-3)/n$. □

Lemma 8. *Let $abcd$ be a convex quadrilateral with $ab = bc$ and $cd < da$. If $\angle adb > \angle bdc$, then $\angle abc + \angle cda < \pi$.*

Proof. Because $cd < da$, c and d lie on the same side of the axis of ac. Let C be the circumcircle of abc. If d lies on C, then $\angle adb = \angle bdc$, because $ab = bc$. If d lies in the interior of C, let A be the circumcircle of cbd and let B be its symmetric with respect to the axis of ac. Because $ab = bc$ and d is internal to C, d is external to B. It follows that $\angle adb < \angle bdc$. Hence d lies in the exterior of C, and therefore $\angle abc + \angle cda < \pi$. □

Lemma 9. *If S is both* Simple biangular *and* Pre-regular*, then two points are neighbors if and only if they are companions.*

Proof. Let $a \in S$ be a point on SEC, let $b \in S$ be the point at angular distance $\mu^{(1)}$ from a, and let $c \in S$ be the point at angular distance $\mu^{(0)}$ from b. If d is the center of SEC, it follows that $\angle adb > \angle bdc$. By Lemma 4, the companion of b is either a or c. Assuming that b's companion is a, we will prove that $\angle abc < \pi(n-3)/n$. This will yield a contradiction by Lemma 7, hence proving our claim.

By Observation 4, $ab \leqslant bc$. If c lies on SEC, then $ab > bc$ (recall that a lies on SEC, as well), a contradiction. If b is fixed and c moves from SEC towards d, $\angle abc$ decreases monotonically. As soon as $ab = bc$, bc starts increasing monotonically. Therefore, it suffices to prove that $ab = bc$ yields $\angle abc < \pi(n-3)/n$. Applying Lemma 8, we get $\angle abc + \angle cda < \pi$. But, since S is biangular, $\angle cda = \mu^{(0)} + \mu^{(1)} = 4\pi/n$, implying that $\angle abc < \pi(n-4)/n < \pi(n-3)/n$. □

Lemma 10. *If S is both* Simple biangular *and* Pre-regular*, and two companions lie on SEC, then every point of S lies on SEC.*

Proof. Let $a, a' \in S$ be two companion points that lie on SEC, which are also neighbors by Lemma 9. Then a and a' are not antipodal, and therefore by Observation 1 there must be another point $b \in S$ on SEC which, without loss of generality, we may assume to be analogous of a. Let p be the center of SEC, and let b' be the neighbor of b, which is also its companion. Because the configuration is *Simple biangular* and the supporting polygon must be regular, it follows that the slope of the line bb' is equal to the slope of aa' increased or decreased by $\angle apb$. Hence also b' lies on SEC.

If the edges of the supporting polygon on which a and b lie are not opposite, then it is easy to see that no two points among a, a', b, b' are antipodal (otherwise S would be *Equiangular*), and they belong to the same half of SEC. By Observation 1, there must be another point $c \in S$ on SEC. By the same reasoning, the companion of c also belongs to SEC. Hence three lines containing edges of the supporting polygon are given, which means that the whole polygon is fixed (by Observation 5), and therefore all the points of S lie on SEC.

Otherwise, if the edges of the supporting polygon on which a and b lie are opposite, the slopes of all other edges are fixed, and the size of the supporting polygon is also fixed. If the center of the polygon is not p, then some points of S must lie outside of SEC, hence the center is p and all the points of S lie on SEC. □

Lemma 11. *If S is both* Simple biangular *and* Pre-regular*, and two non-analogous points lie on SEC, then every point of S lies on SEC.*

Proof. If two points on SEC are neighbors, by Lemma 9 they are also companions, and then Lemma 10 applies. Otherwise, if no two neighbors lie on SEC, by assumption there exist two non-analogous and non-neighboring points $a, b \in S$ that lie on SEC (and belong to different edges of the supporting polygon, by Lemma 9). Let p be the center of SEC. Then, since the supporting polygon is regular, the slope of the edge through b equals the slope of the edge through a plus or minus $\angle apb$. As a consequence, if the companion of a lay strictly inside SEC, then the companion of b would lie strictly outside, which would be a contradiction. Therefore, the companion of a lies on SEC as well, and Lemma 10 applies. □

Lemma 12. *Let S be* Simple biangular*, and suppose that all the points of S that lie on SEC are analogous. If the points of S that are analogous to those on SEC are allowed to "slide" radially toward SEC, then there is at most one configuration of the points that could be* Pre-regular*.*

Proof. By assumption, at least $n/2$ analogous points do not slide, hence no two adjacent points are allowed to slide. Moreover, there is a point $a \in S$ already on SEC that does not slide and, by assumption, neither of its adjacent points is allowed to slide, because they are not analogous to a. Hence Lemma 6 applies. □

Theorem 5. *Let $\mathcal{R}(0)$ be a* Simple biangular *configuration, $n > 4$, and let the robots execute procedure* Simple Biangular *with suitable critical points. Then, the robots eventually reach a* Pre-regular *configuration, and simultaneously stop as soon as they reach it.*

Proof. If $\mathcal{R}(0)$ is already a *Pre-regular* configuration, there is nothing to prove, because none of the robots is moving at time $t = 0$. If two non-analogous points lie on SEC at time $t = 0$, then all robots lie on SEC due to Lemma 11, and therefore the configuration is *Pre-regular*. On the other hand, if all the robots that lie on SEC at time $t = 0$ belong to the same analogy class, procedure Simple Biangular makes the robots of the same analogy class move first toward SEC. By Lemma 12, during this phase at most one configuration C could be *Pre-regular*. Therefore, we may take C as a set of critical points for the cautious move. Note that this set does not change as the robots perform the cautious move. By Corollary 1, the robots simultaneously stop in configuration C, provided that they reach it. If they do not reach it, then by Lemma 1 they eventually reach SEC and stop simultaneously.

Assume now that all the robots of one analogy class are on SEC, forming a *Regular* set of $n/2$ points. Let P be the regular n-gon inscribed in SEC that has these $n/2$ points among its vertices. Procedure Simple Biangular makes the robots of the other analogy class move toward SEC, and the possible *Pre-regular* configurations in which the robots can be found are precisely those in which none of the robots lies strictly in the interior of P, and every two analogous robots are equidistant from the center of SEC.

If all the robots at time $t = 0$ lie inside or on the boundary of P, then we let C be the configuration obtained from $\mathcal{R}(0)$ by sliding all the robots radially away from the center of SEC, until they reach the boundary of P. In this case, the points of C will be the critical points of the cautious move. Otherwise, the cautious move will have no critical points (other than the "final" points on SEC).

If the robots initially lie inside P or on its boundary, the cautious move will make them stop in configuration C, which is the first *Pre-regular* configuration formable. Otherwise, the cautious move will make the robots closest to SEC (and not already on SEC) stay still and wait for their analogous robots to reach the same distance from SEC. During this process no *Pre-regular* configuration is ever reached, until finally all the analogous robots stop at the same distance from SEC, in which case they indeed form a *Pre-regular* configuration. □

4.2.3 Cautious Moves for Periodic Configurations

If S is not co-radial and n is even, we will say that two points of S have *the same parity* (resp. *opposite parity*) if there are an odd (resp. even) number of other points between them in the cyclic order around the center of SEC.

Lemma 13. *If some points of S are allowed to "slide" radially in such a way that SEC never changes, and there are at least four points $a, b, c, d \in S$ that do not slide, appearing in this order around the center of SEC, such that a and b are adjacent, c and d are adjacent,* *and* b *and* c *have the same parity, then there are at most two configurations of the points that could be* Pre-regular*.*

Proof. If some configuration is *Pre-regular*, then by Lemma 4 either a and b are companions and c and d are not, or vice versa. Assume that a and b are companions, and hence the line containing the edge of the supporting polygon through them is fixed. Then the slopes of the two edges through c and d are fixed as well, and this determines a unique supporting polygon, by Observation 5. In turn, this may give rise to at most one possible *Pre-regular* configuration, by Theorem 3. Otherwise, if c and d are companions, by a symmetric argument at most one other *Pre-regular* configuration is formable. □

Observation 6. *If* S *is* Bi-periodic *with period* 3 *and not co-radial, it has exactly two homology classes, one* Equiangular *with* $n/3$ *elements and the other* Simple biangular *with* $2n/3$ *elements.*

Lemma 14. *If* S *is* Bi-periodic *with period* 3*, and the points of the homology class of size* $2n/3$ *are allowed to "slide" radially within SEC while the other* $n/3$ *points stay still on SEC, then no* Pre-regular *configuration is ever formed.*

Proof. By Corollary 2, S is not co-radial, and hence Observation 6 applies. If some *Pre-regular* configuration can be formed, then n must be even, and hence it must be a multiple of 6.

Suppose that $n = 6$. Let $S = \{a, b, c, d, e, f\}$, where the points appear in this order around the center of SEC. Without loss of generality, the angle sequence induced by a is $\{\alpha, \alpha, \beta, \alpha, \alpha, \beta\}$, with $\alpha \neq \beta$. Assume by contradiction that S is *Pre-regular*, let $ABCDEF$ be the supporting polygon, in such a way that a and b lie on edge AB (we may assume this without loss of generality). Let x be the center of the SEC of S and let X be the center of the supporting polygon. Note that e and f must belong to the edge EF, and x lies on the segment be. Therefore, x and A must lie on the same side of the line through B and E. Suppose that $\alpha < 60° < \beta$. Observe that c and d lie on CD and $\angle cxd > 60°$, implying that x lies strictly inside the circle through X, C, and D. However, this circle and A lie on the opposite side of the line though B and E, which yields a contradiction. Assume now that $\alpha > 60° > \beta$. Since $\angle axb > 60°$ and a and b belong to AB, x must lie strictly inside the circle through X, A, and B. Similarly, x must lie strictly inside the circle through X, E, and F. But then x also lies strictly inside the circle through X, F, and A, which contradicts the fact that $a \in AB$, $f \in EF$, and $\angle axf < 60°$.

Suppose now that $n \geqslant 12$. Then, there are an even number of points that stay still on SEC, they are at least four, and they form a *Regular* configuration. Let a one of these points, and a' its antipodal point. Assuming that the points can reach a *Pre-regular* configuration, then a and a' belong to opposite and parallel edges ℓ and ℓ' of the supporting polygon. Therefore, the center of the supporting polygon belongs to the line parallel to ℓ and ℓ' that is equidistant to them. Let r be this line. Since a and a' are antipodal points on SEC, it follows that r passes through the center of SEC. Let b and b' be two antipodal non-sliding points, distinct from a and a'. By the same reasoning, the center of the supporting polygon belongs to a line r' that is parallel to the edges of the supporting polygon through b and b'. Also r' passes through the center of SEC and, since r and r' are not parallel and they are incident at the center of SEC, it follows that the center of the supporting polygon coincides with the center of SEC.

The angle sequence induced by a is of the form $(\alpha, \beta, \alpha, \alpha, \beta, \alpha, \alpha, \beta, \alpha, \cdots)$, with $\alpha \neq \beta$. Observe that, because the period of S is odd, at least two adjacent sliding points must be companions, due to Lemma 4. Hence, Observation 4 implies that $\alpha > \beta$, because the center of the supporting polygon is the center of SEC. It follows that the companion of each sliding point must be another sliding point, which contradicts the fact that the period is odd. □

Lemma 15. *If S is* Double-biangular*, and the points of one homology class stay still on SEC, while the other points are allowed to "slide" radially within SEC, then at most one configuration of the points can be a* Pre-regular *in which sliding points are not companions.*

Proof. Let $p_0 \in S$ be a point belonging to the homology class that stays still on SEC, and let $p_i \in S$ be the $(i+1)$-th point in the cyclic order around the center of SEC, c. We may assume that p_1 is homologous to p_0, and therefore that the angle sequence induced by p_0 is of the form $(\alpha, \beta, \gamma, \beta, \alpha, \beta, \gamma, \beta, \alpha, \beta, \gamma, \beta, \cdots)$. It follows that the points homologous to p_0 are those of the form p_{4i} and p_{4i+1}.

Suppose that S reaches a *Pre-regular* configuration in which no two sliding points are companions. Hence every other edge of the supporting polygon contains a point of S of the homology class that stays still on SEC (cf. Lemma 4). Let q_{2i} (resp. q_{2i+1}) be the point at which the extensions of the edges containing p_{4i} and p_{4i+1} (resp. p_{4i+1} and p_{4i+4}) meet, where indices are taken modulo n. Since the supporting polygon is regular, then clearly the q_i's form a *Regular* configuration with $n/2$ elements, and in particular $q_0q_1 = q_1q_2$ and $\angle p_0q_0p_1 = \angle p_1q_1p_4 = \angle p_4q_2p_5 = \pi(n-4)/n$. On the other hand, the homology class of p_0 is a *Simple biangular* or *Equiangular* set of size $n/2$ lying on SEC, hence it forms a polygon with equal internal angles, and in particular $\angle p_0p_1p_4 = \angle p_1p_4p_5 = \pi(n-4)/n$.

Let $\theta = \angle p_1p_0q_0$ and $\theta' = \angle q_0p_1p_0$. Then

$$\pi - \theta - \theta' = \angle p_0q_0p_1 = \pi(n-4)/n = \angle p_0p_1p_4 = \pi - \theta' - \angle p_4p_1q_1,$$

implying that $\angle p_4p_1q_1 = \theta$. Analogously $\angle p_5p_4q_2 = \theta$, and therefore $p_0p_1q_0$ and $p_1p_4q_1$ are similar triangles, and $p_0p_1q_0$ and $p_4p_5q_2$ are congruent (because $p_0p_1 = p_4p_5$).

We have $q_0p_1 + p_1q_1 = q_0q_1 = q_1q_2 = q_1p_4 + p_4q_2$. Also, $p_0q_0/p_1q_1 = q_0p_1/q_1p_4$ and $p_0q_0 = p_4q_2$, hence we may substitute q_1p_4 with $q_0p_1 \cdot p_1q_1/p_0q_0$ and p_4q_2 with p_0q_0. After some algebraic manipulations, we obtain $(p_0q_0 - p_1q_1)(p_0q_0 - q_0p_1) = 0$, which implies that either $p_0q_0 = p_1q_1$ or $p_0q_0 = q_0p_1$.

Assume first that $p_0q_0 = p_1q_1$ and $p_0q_0 \neq q_0p_1$. This implies that $\alpha = 2\beta + \gamma = 4\pi/n$ and therefore, by observing that the sum of the internal angles of the quadrilateral $cp_1q_1p_4$ is 2π, we have $\angle cp_1q_1 = \pi - \angle q_1p_4c$. This means that the segment p_1q_1 has some points inside SEC if and only if q_1p_4 has none. However, p_2 is the companion of p_1 and hence it lies on p_1q_1, and p_3 is the companion of p_4 and hence it lies on q_1p_4, which yields a contradiction. It follows that in this case no *Pre-regular* configuration is formable.

Assume now that $p_0q_0 = q_0p_1$, hence $\angle cp_0q_0 = \angle q_0p_1c = \pi(n+4)/2n - \alpha/2$. Therefore the slopes of the two edges of the supporting polygon to which p_0 and p_1 belong are fixed. This also fixes the slope of the edge of the supporting polygon through p_4, and hence the whole supporting polygon is fixed, by Observation 5. Due to Theorem 3, the trajectory of each sliding point intersects the supporting polygon in at most one point, and therefore in this case at most one *Pre-regular* configuration can be formed. □

Lemma 16. *Let* $\mathcal{R}(0)$ *be a* Double-biangular *configuration, and let the robots of one homology class stay still on SEC, while the other robots move radially, either all toward SEC or all toward SEC/4, performing a cautious move with suitable critical points. Then, as soon as the robots reach a* Pre-regular *configuration in which moving robots are companions, they simultaneously stop.*

Proof. Suppose first that $n \geqslant 12$. If $\mathcal{R}(t)$ is *Pre-regular* at some time $t \geqslant 0$, there are at least three pairs of companions that stay still on SEC (cf. Lemma 4). These three pairs determine the slopes of three edges of the supporting polygon, which, due to Observation 5, is fixed. By Theorem 3, the trajectory of each robot intersects the supporting polygon in at most one point, and hence there is at most one formable *Pre-regular* configuration, which can be chosen as a set of critical points for the cautious move, due to Theorem 1.

Let $n < 12$, and hence $n = 8$. Let $\mathcal{R} = \{a, b, c, d, e, f, g, h\}$, where c, d, g, and h stay still on SEC. We seek to characterize the formable *Pre-regular* configurations in which c and d are companions. Let ℓ be the line through c and d, let ℓ' be the line through g and h, and let λ be the distance between ℓ and ℓ'. Then, the two edges of the supporting polygon to which a and b belong must be orthogonal to both ℓ and ℓ', and similarly for the edge to which e and f belong. Moreover, the distance between these two edges must be λ. Let x be the center of SEC, and let a' (resp. b', e', f') be the point on SEC that is co-radial with a (resp. b, e, f). It is easy to see that the positions of a that could give rise to a *Pre-regular* configuration belong to a (possibly empty) closed segment A, which is a subset of the segment $a'x$. Similarly, the positions of b, e, and f that could give rise to *Pre-regular* configurations belong to closed segments B, E, and F, which, together with A, form a set that is mirror symmetric and centrally symmetric with respect to x. If A is empty, then no *Pre-regular* configuration in which moving robots are companions can be formed. Therefore, let us assume that A is not empty.

We assume that a, b, e, and f all move toward SEC. The case in which they all move toward x is symmetric, and therefore it is omitted. Let a'' and a''' be the endpoints of A, with a'' closest to a', and let a^* be the midpoint of A. Analogous names are given to the endpoints and midpoints of B, E, and F. Note that, by construction, $\{a'', b'', c(t), d(t), e''', f''', g(t), h(t)\}$, $\{a''', b''', c(t), d(t), e'', f'', g(t), h(t)\}$, and $\{a^*, b^*, c(t), d(t), e^*, f^*, g(t), h(t)\}$ are *Pre-regular* sets.

Without loss of generality, let $a(0)$ be such that $a(0)a'$ is not longer than $b(0)b'$, $e(0)e'$, and $f(0)f'$. If $a(0)$ belongs to the segment $a''a'$, open at a'' and closed at a', then no *Pre-regular* configuration can be formed, regardless of how the robots move toward SEC. Hence in this case no critical points are needed. If $a(0)$ belongs to the (closed) segment xa^*, then we take $\{a^*, b^*, c(t), d(t), e^*, f^*, g(t), h(t)\}$ as a set of critical points. Since $b(0) \in xb^*$, $e(0) \in xe^*$, and $f(0) \in xf^*$, by the cautious move protocol a, b, e, and f will stop at a^*, b^*, e^*, and f^*, respectively, and wait for each other. When all of them have reached the critical points, a *Pre-regular* configuration is reached, and none of the robots is moving. Also, this is the first *Pre-regular* configuration that is reached by the robots.

Finally, let $a(0)$ belong to the segment a^*a'', open at a^* and closed at a''. Let b_1 and b_2 be the two points on xb' whose distance from b^* is the same as the distance between $a(0)$

and a^*, with b_1 closest to x. Analogously, we define e_1 and e_2 on xe', and f_1 and f_2 on xf'. Then, the set $\{a(t), b_2, c(t), d(t), e_1, f_1, g(t), h(t)\}$ is *Pre-regular*, and we may take it as a set of critical points. If $e(0)$ is past e_1, or $f(0)$ is past f_1, then no *Pre-regular* set can be formed, regardless of how the robots move. Otherwise, the cautious move protocol will make e and f reach e_1 and f_1, stop there, and wait for each other. Note that (note that $a(t)$ does not change while this happens).

If $b(0) = b_2$, then a *Pre-regular* configuration is reached for the first time, and none of the robots is moving. Otherwise, suppose that $b(0)$ is in the (closed) segment xb_1. Then, eventually, b will stop in b_1 at the same time as e and f are in e_1 and f_1. Next, e and f will acquire e_2 and f_2 as new critical points (because b_2 is a critical point of b), and also e^* and f^* (because they are the midpoints of e_1e_2 and f_1f_2). Similarly, b will acquire b^* as a new critical point. When all three of them have moved once, they will be found somewhere in the *open* segments b_1b_2, e_1e_2, and f_1f_2. While they reach this configuration, no *Pre-regular* configuration is ever formed. Moreover, no *Pre-regular* configuration can be formed afterwards. Finally, let $b(0)$ be in the open segment b_1b_2. Then, b will stay still and wait for e and f, which will eventually move and stop somewhere in the *open* segments e_1e_2 and f_1f_2. As in the previous case, no *Pre-regular* configuration can ever be reached. □

Theorem 6. *Let $\mathcal{R}(0)$ be a* Periodic *configuration with period $k > 2$, $n > 4$, and let the robots execute procedure* PERIODIC *with suitable critical points. Then, as soon as the robots reach a* Pre-regular *configuration, they simultaneously stop.*

Proof. In procedure PERIODIC, the only moves that the robots make are radial, either toward SEC or toward its center. Therefore, all co-radialities between robots are preserved throughout this procedure. Due to Corollary 2, if some robots are co-radial, the configuration may never become *Pre-regular*, and therefore we may assume that $\mathcal{R}(0)$ is not co-radial. By definition of period, $n \geqslant 2k$. Recall that, at each phase of the procedure, only homologous robots are allowed to move simultaneously. When the referees reach SEC, they stop and another homology class of robots starts moving toward SEC/4. As soon as the first robot has reached SEC/4, the procedure keeps it on or inside SEC/4. Therefore, from this point on, no *Pre-regular* can be formed, due to Theorem 4. As a consequence we may assume that, if $\mathcal{R}$ can form a *Pre-regular* configuration, only one homology class is moving. Let $r_0 \in \mathcal{R}$ be a moving robot, and let $r_i \in \mathcal{R}$ be the $(i+1)$-th robot in the cyclic order around the center of SEC. By definition of homology class, in every set of k consecutive robots (in their cyclic order around the center of SEC), at most two of them are allowed to move.

Suppose first that exactly n/k robots are allowed to move. If $k > 3$, then r_1, r_2, r_3, and r_{k+1} do not move, and they satisfy the hypotheses of Lemma 6. If $k = 3$, then Lemma 13 applies to r_1, r_2, r_4, and r_5, and at most one configuration C can be *Pre-regular*. By Theorem 1, taking C as a set of critical points suffices.

Otherwise $2n/k$ robots are allowed to move, and therefore $\mathcal{R}(0)$ is *Bi-periodic*. Let r_a be a moving robot such that $0 < a < k$. Without loss of generality, we may assume that $a \leqslant \lfloor k/2 \rfloor$. If $k > 6$, then Lemma 6 applies to r_{k-3}, r_{k-2}, r_{k-1}, and r_{2k-1}, which do not move. If $k = 5$, then Lemma 13 applies to r_3, r_4, r_8, and r_9. If $k = 6$ and $a = 1$ or $a = 2$, then Lemma 6 applies to r_3, r_4, r_5, and r_{11}. If $k = 6$ and $a = 3$, then Lemma 13 applies to r_1, r_2, r_4, and r_5. Hence at most two configurations can be *Pre-regular*. By Corollary 1, taking the union of these configurations as critical points suffices.

Only three cases are left.

- Let $k = 3$. By Observation 6, there are only two homology classes: one with $n/3$ robots, which are the referees, and the other with $2n/3$ points. According to the procedure, if the non-referees are allowed to move, it means that the referees are located on SEC. Therefore Lemma 14 applies, and no *Pre-regular* configuration can be formed.

- Let $k = 4$ and $a = 2$. Then, r_1 and r_3 are necessarily non-homologous, and hence they belong to distinct homology classes, each of size n/k. Therefore, one of these classes contains the referees, which stay still on SEC. The other class, according to the procedure, moves to SEC/4 before any other homology class. It follows that, if $2n/k$ robots are allowed to move, then n/k robots must be already on or inside SEC/4, implying that no *Pre-regular* configuration can be formed, due to Theorem 4.

- Let $k = 4$ and $a = 1$. Then $\mathcal{R}(0)$ is *Double-biangular*. According to the procedure, three things may happen.

  - If none of the referees is on SEC, then the referees stay still and the non-referees move radially to SEC. This means that some non-referees are already on SEC, and therefore they also stay still. The existence of a non-referee that stays still implies the presence of three consecutive robots that do not slide, and enables the application of Lemma 6. Hence at most one *Pre-regular* configuration is formable, which can be taken as a set of critical points, due to Theorem 1.
  - If some of the referees are not on SEC and all the non-referees are on SEC, the referees move radially to SEC. By Lemma 15, at most one *Pre-regular* configuration C_1 is formable in which moving robots are not companions. Theorem 1 guarantees that the cautious move with critical point set C_1 enjoys property $\mathcal{P}_1$ that the robots simultaneously stop as soon as they reach configuration C_1. On the other hand, by Lemma 16, there exists a set of critical points C_2 ensuring property $\mathcal{P}_2$ that the robots will simultaneously stop in a *Pre-regular* configuration in which moving robots are companions, if one is ever reached. Hence, due to Theorem 2, the cautious move with critical point set $C_1 \cup C_2$ enjoys both properties $\mathcal{P}_1$ and $\mathcal{P}_2$, and therefore it correctly handles all formable *Pre-regular* configurations.
  - If all the referees are on SEC, the non-referees move radially toward the center of SEC. The analysis of the previous paragraph applies verbatim to this case.

□

4.2.4 Cautious Moves for Aperiodic Configurations

Lemma 17. *Let $\mathcal{R}(0)$ be a non-co-radial* Aperiodic *configuration, $n = 6$, and let the robots of one homology class move radially, either all toward SEC or all toward SEC/3, in such a way that SEC does not change, and performing a cautious move with suitable critical points. Then, as soon as the robots reach a* Pre-regular *configuration, they simultaneously stop.*

Proof. Recall that, in an *Aperiodic* configuration, the homology classes have size either one or two. If there is only one robot moving, then Lemma 6 applies, and at most one *Pre-regular* configuration C can be formed. Taking C as a set of critical points suffices, due to Theorem 1.

Suppose that exactly two robots are moving. Let r_0 be a moving robot, and let r_i be the i-th robot in the cyclic order around the center of SEC. Without loss of generality, either r_1 or r_2 or r_3 is moving. If r_2 is moving, then at most one *Pre-regular* configuration C is formable, due to Lemma 6. C can be taken as a set of critical points, due to Theorem 1. On the other hand, if r_3 is moving, Lemma 13 applies, and at most two *Pre-regular* configurations C_1 and C_2 can be formed. Therefore, by Corollary 1, taking $C_1 \cup C_2$ as a set of critical points suffices.

Finally, assume that r_1 is moving, along with r_0. If r_0 and r_1 are not companions, then by Lemma 4 r_3 and r_4 are. Hence the slope of the edge of the supporting polygon through r_3 and r_4 is fixed, which fixes also the slopes of the edges through r_2 and r_5. Hence, by Observation 5, the whole supporting polygon is fixed, implying that at most one configuration C of the robots can be *Pre-regular*, due to Theorem 3. Taking C as a set of critical points suffices for all *Pre-regular* configurations in which r_0 and r_1 are not companions, due to Theorem 1.

In the following, we will assume that r_0 and r_1 are companions. Suppose first that both r_0 and r_1 are moving toward SEC. By Lemma 4, r_2 and r_3 are companions, and they determine the slope of one edge of the supporting polygon. Therefore, the slope of the edge containing r_0 and r_1 is also fixed. Let x be the center of the sec of $\mathcal{R}(0)$ (which, by assumption, does not change over time), and let us consider the two rays from x and through $r_0(0)$ and $r_1(0)$. Let f_0 and f_1 be, respectively, the points at which these two rays intersect the perimeter of SEC. As r_0 and r_1 move radially between x and the perimeter of SEC, they can conceivably form infinitely many *Pre-regular* configurations. However, the positions of r_0 on xf_0 that could give rise to *Pre-regular* configurations form a closed interval aa', with a closest to x (we assume this interval to be non-empty, otherwise we may take $C' = \varnothing$ as the set of critical points). Similarly, the positions of r_1 on xf_1 giving rise to *Pre-regular* configurations identify a closed interval bb', with b closest to x. Moreover, the line ℓ through a and b and the line ℓ' through a' and b' are parallel, because the slope of the edge of the supporting polygon containing r_0 and r_1 is fixed.

Suppose first that ℓ is parallel to the line through f_1 and f_2. Equivalently, xa and xb have the same length. In this case, we take $C' = \{a, b\}$ as a set of critical points. Indeed, let us assume without loss of generality that $r_0(0)f_0$ is not longer than $r_1(0)f_1$. If $r_0(0)$ is past a', no *Pre-regular* can ever be formed, and there is nothing to prove. If $r_0(0)$ lies on the closed segment aa', the cautious move protocol will make r_0 stay still and wait for r_1 to reach the same distance from the endpoint of its respective path, and stop. When this happens, say at time t, the line through $r_0(t)$ and $r_1(t)$ is parallel to ℓ, and therefore the configuration is *Pre-regular*. Moreover, no *Pre-regular* configuration is reached before time t. Finally, let $r_0(0)$ be before a. Then, the cautious move protocol makes r_0 and r_1 stop at a and b respectively, and wait for each other. When the robots reach a and b, the configuration is the first *Pre-regular* encountered.

Suppose now that ℓ is not parallel to the line through f_1 and f_2. Without loss of generality, suppose that xa is longer than xb. First of all, if $r_0(0)$ is located past a' or $r_1(0)$ is located past b', no *Pre-regular* configuration can be formed, and there is nothing to prove. Let $r_0(0)$ belong to the closed segment aa', and let c be the point on bb' such that the line through $r_0(0)$ and c is parallel to ℓ. If $r_1(0)$ lies not after c, we take $C' = \{c\}$ as a set of critical points. Indeed, the cautious move protocol makes r_0 stay still and wait for r_1 to reach c and stop. When this happens the configuration is *Pre-regular*, and no other *Pre-regular* configuration is reached before.

Now assume that $r_1(0)$ lies after c (as defined above), or that $r_0(0)$ lies before a. We let $c_0 = b$ and we let c_1 be the intersection between bf_1 and the line through a and parallel to f_0f_1. Then we inductively define c_{i+2}, with $i \geqslant 0$, to be the point on bf_1 such that the length of xc_{i+1} is the geometric mean between the lengths of xc_i and xc_{i+2}. Let k be the highest index such that c_k is well defined, and let $c_{k+1} = f_1$. For each $0 \leqslant i \leqslant k+1$, we define ℓ_i to be the line through c_i and parallel to f_0f_1. Then, we let L_i be the region of the plane that lies between lines ℓ_i and ℓ_{i+1}, such that $\ell_i \subset L_i$ and $L_i \cap \ell_{i+1} = \varnothing$ (unless $\ell_k = \ell_{k+1}$, in which case $L_k = \ell_k$). We argue that taking $C' = \{c_0, \cdots, c_{k+1}\}$ as a set of critical points prevents the robots from reaching any *Pre-regular* configuration during the cautious move. Note that a *Pre-regular* configuration can be formed at time t only if $r_1(t) \in L_i$ and $r_0(t) \in L_{i+1}$, for some $0 \leqslant i \leqslant k-1$. This can be true at time $t = 0$ but, due to our assumptions, it implies that $r_1(0)$ lies after c, and hence r_1 will reach L_{i+1} while r_0 waits, without forming a *Pre-regular* configuration. Similarly, if both robots lie initially before L_0, the cautious move protocol will make them reach L_0 and wait for each other before proceeding. Moreover, if $r_0(t) \in L_i$ and $r_1(t) \in L_j$ with $j > i$, then r_1 waits until r_0 reaches L_j, and during this process no *Pre-regular* configuration is formed.

Therefore we can assume that, at some time t, both $r_0(t)$ and $r_1(t)$ belong to L_i, for some $0 \leqslant i \leqslant k$, and none of them is moving. We claim that, if $i < k$, there is a time $t' > t$ at which the two robots are in L_{i+1} and none of them is moving. Moreover, between t and t' no *Pre-regular* configuration is reached. Indeed, due to the cautious move protocol, the robots stop at ℓ_{i+1} and wait for each other before proceeding, and hence at some point they will clearly be found both in L_{i+1} and not moving. The only way they could form a *Pre-regular* configuration would be if r_0 reached ℓ_{i+1} when r_1 was still at ℓ_i. But this cannot happen because, due to the cautious move protocol, r_0 stops and at least once after ℓ_i and before ℓ_{i+1}. When this happens, r_0 cannot proceed any further, and hence it cannot reach ℓ_{i+1} if r_1 is still at ℓ_i. By induction on i, it follows that r_0 and r_1 eventually reach f_0 and f_1, respectively, without ever forming a *Pre-regular* configuration.

Finally, let us consider the case in which r_0 and r_1 move toward SEC/3. If one of the two robots is initially on or inside SEC/3, by Theorem 4 no *Pre-regular* configuration can ever be formed. Hence we may assume that both robots move radially inwards with destinations on SEC/3. This case is symmetric to the previous one, and can be treated with a similar reasoning.

To conclude, taking $C \cup C'$ as a set of critical points yields a cautious move that makes the robots stop at every *Pre-regular* configuration that is encountered (i.e., whether r_0 and r_1 are companions or not), due to Theorem 2. □

Theorem 7. *Let $\mathcal{R}(0)$ be an* Aperiodic *configuration, $n > 4$, and let the robots execute procedure* APERIODIC. *Then, as soon as they reach a* Pre-regular *configuration, they simultaneously stop.*

Proof. In procedure APERIODIC, the only moves that the robots make are radial, either toward SEC or toward SEC/3. Therefore, all co-radialities between robots are preserved throughout this procedure. Due to Corollary 2, if some robots are co-radial, the configuration may never become *Pre-regular*, and therefore we may assume that $\mathcal{R}(0)$ is not co-radial. Moreover, recall that n must be even for a *Pre-regular* configuration to be formed. If $n = 6$, Lemma 17 applies. Hence, in the following we will assume that $n \geqslant 8$.

In the first phase, homology classes of robots take turns moving toward SEC. Because there are no co-radialities, all moving robots are able to reach SEC. Therefore, a new homology class of robots starts moving only when the previous one has actually reached SEC, implying that at any time only homologous robots are moving.

Next we prove that, as a homology class of robots moves radially without altering SEC, then at most one *Pre-regular* configuration can be formed. Recall that the homology classes of an *Aperiodic* configuration contain either one or two points; it follows that, no matter how one or two robots are chosen to slide, the hypotheses of Lemma 6 are satisfied, and therefore at most one *Pre-regular* configuration can be formed. If such a configuration is taken as a set of critical points, our claim follows from Theorem 1.

The same reasoning applies also to the second phase of procedure APERIODIC, where the referees move radially toward SEC/3. □

Remark 1. *It is easy to verify that, in all the above theorems, the critical points of the cautious moves are computable by performing finite sequences of algebraic operations (i.e., arithmetic operations plus taking roots) on the positions of the robots.*

4.3 Correctness of the Algorithm

The correctness of the algorithm follows from the fact that the directed graph of configurations and their transitions contains no cycles, and the only sink is the *Regular* configuration (see Figure 4).

To verify that indeed no other transitions are possible, several elements are needed. The first step is to complete the definition of every configuration with the destination point that each robot may have. This is necessary, due to the asynchronous nature of the robots. The possible positions of the robots, together with their possible destination points, constitute the *invariant* of a configuration. Below we list the invariants of all configurations.

***Regular*:**

- All robots are still.

***Pre-regular*:**

- Each robot is moving towards its *matching vertex* of the supporting polygon.

***Central*:** No configuration becomes *Central* (i.e., this can just be the initial configuration), thus the invariant here is irrelevant.

***Equiangular*:**

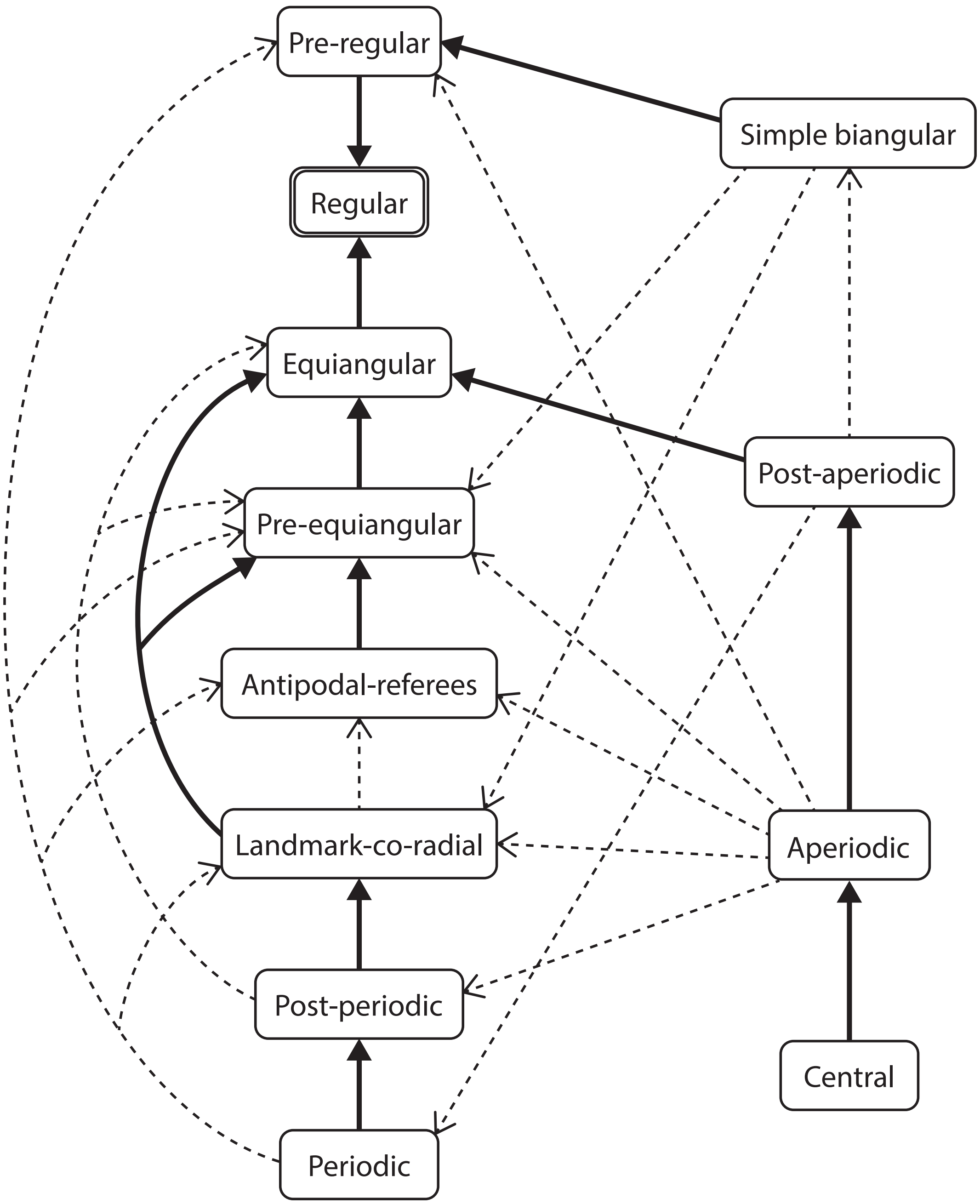


Figure 4: Configurations, with their intended transitions (thick arrows) and incidental transitions (dashed arrows)

- Each robot is moving radially towards SEC.

Simple biangular:

- Each robot is doing a cautious move radially outwards.

Periodic: The robots can reach this configuration only after *Post-aperiodic*, as a critical point of a cautious move; hence, with all robots still. This makes the invariant for this case irrelevant.

Aperiodic:

- All robots are moving radially outwards using a cautious move whose first critical point is on SEC/4.
- The referees are moving radially towards SEC/3 with a cautious move whose most internal critical point is on SEC/4.

Post-periodic: Note that this configuration is reached only from a *Periodic* or *Aperiodic*; in particular, from *Aperiodic* it can be reached only as a critical point, thus with all robots still.

- The non-referees internal to the sectors defined by the landmarks are moving radially on mutually disjoint trajectories.
- The non-referees that are co-radial with the landmarks are moving radially inwards, on mutually disjoint trajectories.
- Some robots are rotating (this is irrelevant, since in *Periodic* no robot can rotate).

Landmark-co-radial:

- The robots that are co-radial with the referees are moving radially inwards on mutually disjoint trajectories.
- The walkers are rotating inside SEC/4, away from a referee and towards their targets.
- If all the non-referees are on their targets and the configuration is *Antipodal-referees*, the robots closest to the referees are moving towards SEC.
- All other robots are still.

Antipodal-referees:

- The two robots that are angularly closest to the referees are moving radially from SEC/4 to SEC.

Pre-equiangular:

- The robots on SEC are rotating towards their targets.

***Post-aperiodic*:** This configuration is reached only from *Aperiodic* when all robots are still; hence, the invariant is irrelevant.

Because the algorithm forces the robots to move in a strictly controlled way, synchronizing their actions whenever possible, the invariants end up being quite simple in most cases. This makes every single test for consistency straightforward, although testing all possible transitions is still a lengthy process.

Lemma 18. *The above invariants are preserved throughout the execution of the algorithm.*

Proof. It is easy to check that, whenever a transition from a configuration X to a configuration Y occurs, all the robots satisfy the invariant of Y, assuming that they already satisfy the invariant of X and considering the actions that the robots are currently performing, according to the procedure that is executed in X. □

The order in which the different configurations are tested by the algorithm also plays an important role. It has been chosen so to minimize the possible conflicts between configurations, and to give them a consistent "flow" towards the *Regular* configuration.

The next step is to prove that no "unwanted" transition is possible.

Lemma 19. *If $n > 4$, no transition is possible other than those illustrated in Figure 4.*

Proof. First of all, we observe that the initial SEC of the robots never changes (unless the configuration becomes *Pre-regular*), although the robots that lie on its boundary and "define" it may move, or change.

To prove that most transitions are not possible, it is sufficient to keep in mind that the robots occupy specific circles in each configuration. For instance, in a *Post-aperiodic* configuration all robots lie outside of SEC/4, and therefore the configuration cannot be (or transition to) *Central*, *Post-periodic*, *Landmark-co-radial*, *Antipodal-referees*, or *Pre-equiangular*, because all of these must contain robots on or inside SEC/4. For the same reason, the vice versa also holds.

For the *Pre-regular* case a more sophisticated analysis is needed. Two important results are that no *Pre-regular* configuration can have co-radial robots or robots on or inside SEC/3 (see Corollary 2 and 4 in Section 4.2.1). As a consequence, the only configurations that may "incidentally" be *Pre-regular* are the *Equiangular*, *Simple biangular*, *Periodic*, and *Aperiodic*.

Finally, the results in Section 4.1 show that the cautious move protocol is sound. Hence, no problem arises when the critical points that are needed for the cautious moves are trivially finitely many, as the points on SEC/4 or on SEC/3 that lie on a robot's trajectory, or the the points that may give rise to a *Periodic* or a *Simple biangular* configuration, etc. On the other hand, the *Pre-regular* case is harder to analyze, and Section 4.2 is dedicated to showing that there are finite sets of critical points that correctly handle these configurations, as well (see Theorems 5, 6, and 7). □

From the previous lemmas, the correctness of the algorithm straightforwardly follows.

Theorem 8. *The* Uniform Circle Formation *problem is solvable by $n \neq 4$ robots in $\mathcal{ADO}$.*

Proof. The algorithm in Figure 1 solves the Uniform Circle Formation problem for $n > 4$ robots, due to Lemmas 18 and 19.

If $n < 3$, the problem is trivial. If $n = 3$, we use the following ad-hoc algorithm.

- Let the robots form a scalene triangle, whose largest edge has robots r_1 and r_2 as endpoints. Then, robot r_3 moves parallel to r_1r_2 until it reaches the axis of r_1r_2.
- Let the robots form an isosceles triangle with $r_1r_2 = r_1r_3$. Then, r_1 moves to the point that forms an equilateral triangle with r_2 and r_3.

□

The case $n = 4$ is still open. As a final note we remark that, at the cost of slightly modifying our algorithm and analysis, it is possible to eliminate all circular movements, and let the robots move only along straight lines.

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