

Minimal Arrangements of Spherical Geodesics

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Abstract

We study arrangements of geodesic arcs on a sphere, where all arcs are internally disjoint and each arc has its endpoints located within the interior of other arcs. We establish fundamental results concerning the minimum number of arcs in such arrangements, depending on local geometric constraints such as *one-sidedness* and *k-orientation*.

En route to these results, we generalize and settle an open problem from CCCG 2022, proving that any such arrangement has at least two *clockwise swirls* and at least two *counterclockwise swirls*.

2012 ACM Subject Classification Mathematics of computing → Geometric topology; Mathematics of computing → Extremal graph theory

Keywords and phrases Minimal arrangement, spherical geodesic, visibility map, polyhedron, swirl graph, one-sided, *k*-oriented

1 Introduction

1.1 Art Gallery Problem and Spherical Diagrams

In Discrete and Computational Geometry, the *Art Gallery Problem* asks how many “light sources” are required to “illuminate” a given geometric environment [7, 9]. The environment may be an enclosure, such as a polygon or a polyhedron, or more broadly, a spatial arrangement of “opaque” objects that obstruct light rays.

In recent years, a line of research on the Art Gallery Problem has focused on illuminating 3-dimensional polyhedra by choosing a subset of their edges as light sources. In this setting, the facets of a polyhedron are interpreted as opaque obstacles, and one wants to illuminate the polyhedron’s interior with as few “edge lights” as possible [1, 2, 4, 11].

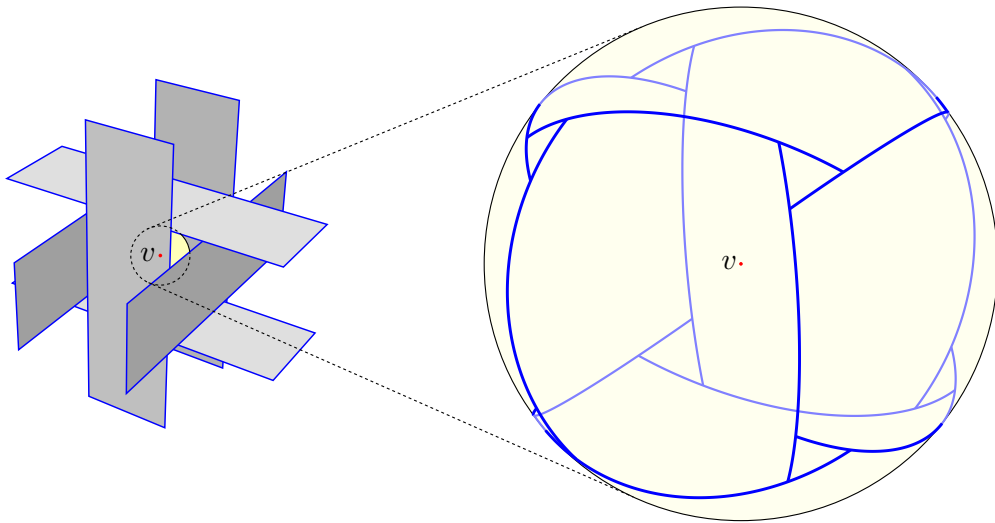
To this end, a technique developed in [4] involves selecting a small set of edges whose endpoints collectively include all vertices of the polyhedron. Interestingly, however, these edges may not be sufficient to fully illuminate the polyhedron’s interior, as there is an abundance of polyhedra having internal points that do not directly see any vertices [7, 12, 13]. We say that such points are *vertex-hidden*. Thus, the study of vertex-hidden points and the combinatorics of their visibility maps becomes central in [4], as well as in the development of a general theory of visibility-related problems for polyhedral objects in 3-space [8].

In order to systematize these fundamental investigations, Spherical Occlusion Diagrams (SODs) were introduced as a model for visibility maps of vertex-hidden viewpoints relative to *polygonal scenes*, i.e., arrangements of interior-disjoint polygons in 3-space [12, 13].

We give some preliminary definitions. A *geodesic arc* on a sphere is the unique shortest curve connecting two non-antipodal points. Clearly, any geodesic arc lies within a great circle on the sphere. We say that a geodesic arc *a* *blocks* a geodesic arc *b* if an endpoint of *b* lies in the relative interior of *a*. If *a* blocks *b*, we say that *b* *hits* *a*.

► **Definition 1.** A Spherical Diagram (SD) is a finite non-empty collection \mathcal{D} of pairwise interior-disjoint geodesic arcs on the unit sphere in \mathbb{R}^3 , such that each arc of \mathcal{D} is blocked by arcs of \mathcal{D} at each endpoint.

► **Definition 2.** A Spherical Occlusion Diagram (SOD) is a Spherical Diagram \mathcal{D} with the additional property that all arcs in \mathcal{D} that hit the same arc of \mathcal{D} reach it from the same side.



■ **Figure 1** A polygonal scene of six rectangles (left), where the central point v is vertex-hidden. The portion of each edge that is visible to v is radially projected onto a sphere centered at v , creating a geodesic arc. The resulting visibility map is an SOD (right).

SDs may also be referred to as *two-sided* arrangements of arcs, whereas SODs are *one-sided*, to stress the fact that their arcs are hit only from one side (the term “one-sided” was previously used in [5] in the context of rectangular layouts). While SODs were introduced and studied in [8, 12, 13], in this paper we also set out to investigate SDs as a natural generalization. Notably, most of what is currently known about SODs is more generally true for SDs (refer to Section 2.1), and SDs are interesting geometric objects in their own right.

As illustrated in Figure 1, the visibility map of a vertex-hidden viewpoint in a polygonal scene is necessarily an SOD (see [13]). Interestingly, the converse is not always the case, as was shown in [6]. Nonetheless, SODs were instrumental in proving that any vertex-hidden viewpoint in a polygonal scene must see at least eight distinct edges (this number reduces to six for viewpoints that are not vertex-hidden). This result was obtained in [8] by showing that any SOD consists of at least eight arcs. In turn, this follows from the observation that any SOD (in fact, any SD) must have at least four *swirls*, which are defined next.

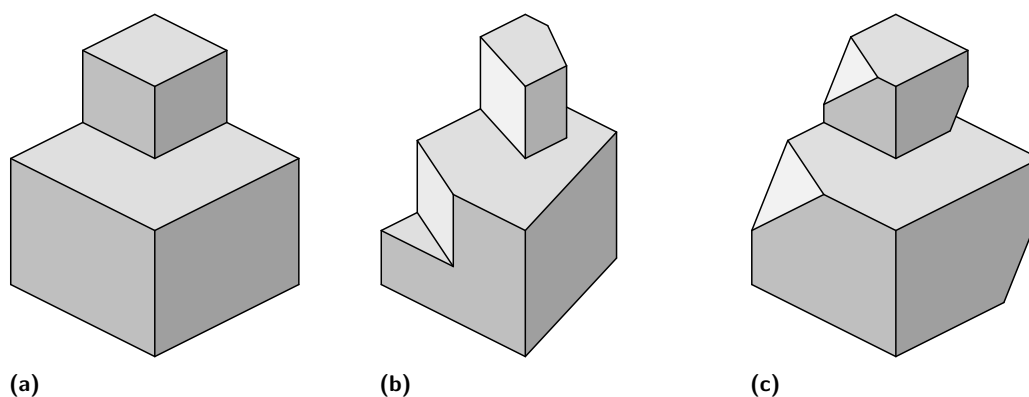
► **Definition 3.** A swirl in an SD is a cycle of arcs, each of which hits the next (and such that the last hits the first), going either all clockwise or all counterclockwise. The degree of a swirl is the number of arcs constituting it.

As an example, the SOD in Figure 1 has four clockwise swirls and four counterclockwise swirls, all of degree three.

1.2 Polyhedra with Restricted Edge Orientations

In the study of the Art Gallery Problem, it is customary to investigate not only polygons or polyhedra in their full generality, but also specific subclasses with particular geometric properties. For example, alongside general polyhedra, also *orthogonal polyhedra* have been considered (Figure 2a): these are polyhedra whose faces meet at right angles [1, 2, 11].

Similarly, a *k-face-oriented* polyhedron has the property that there exist k unit vectors such that the normal of each face is parallel to one of these vectors [10]. Note that orthogonal polyhedra are 3-face-oriented, and any polyhedron is k -face-oriented for a large-enough k .



■ **Figure 2** (a) An orthogonal polyhedron. (b) A 7-edge-oriented polyhedron having only vertical and horizontal edges (and faces). (c) A 4-face-oriented and 6-edge-oriented polyhedron.

In the same vein, we may define a polygonal scene to be *k*-edge-oriented if there exist *k* unit vectors such that every edge of every polygon in the scene is parallel to one of the vectors. Again, any orthogonal polyhedron is a 3-edge-oriented polygonal scene; in general, any *k*-face-oriented polyhedron is also a $\binom{k}{2}$ -edge-oriented polygonal scene (Figure 2c).

Following the seminal result of [8] mentioned in Section 1.1, our motivating question is: **What is the minimum number of edges that are necessarily visible to a vertex-hidden viewpoint in a *k*-edge-oriented polygonal scene?** In order to model the visibility maps of such viewpoints, we give the following definition.

► **Definition 4.** An SD \mathcal{D} is *k*-oriented if there exists a set P of *k* points on the unit sphere, called poles, no two of which are antipodal, as well as a function $f: \mathcal{D} \rightarrow P$ such that each arc $a \in \mathcal{D}$ is collinear with the pole $f(a)$, but contains neither $f(a)$ nor its antipodal point.

With the notation of Definition 4, we refer to the point antipodal to a pole $p \in P$ as the *anti-pole* of p . Also, for any arc $a \in \mathcal{D}$, the pole $f(a)$ and its anti-pole $-f(a)$ are said to be the *vanishing points* of a .¹ If three of the *k* poles lie on a same great circle, then \mathcal{D} is said to be *degenerate*. Otherwise, \mathcal{D} is *non-degenerate*.

It is immediate to recognize that the visibility map of any vertex-hidden point of a *k*-edge-oriented polygonal scene is a *k*-oriented SOD. The converse is not true in general (a counterexample is found in [6]), but it may be true for sufficiently small values of *k*.

Thus, we recast our motivating question: **What is the minimum number of arcs that a *k*-oriented SD or SOD can have?** While this formulation may not be strictly equivalent to the one about polygonal scenes, it stands as a question of independent interest in extremal graph theory [3], because SDs can be characterized as spherical drawings of certain graphs. Consider a non-empty 3-regular planar graph G that can be drawn on a sphere by means of internally disjoint geodesic arcs in such a way that each vertex of G is incident to two collinear arcs, and any chain of such collinear arcs is itself a geodesic arc. Such a drawing coincides with an SD in which no two arcs have a common endpoint, and vice versa. For the purpose of minimizing arcs, this is not a restrictive assumption on SDs; also, it is easy to see that minimizing arcs in SDs is equivalent to minimizing vertices (or edges) in such graphs. *k*-oriented SDs and SODs can be characterized in similar ways, as well.

¹ The term “vanishing point” comes from perspective theory, and indicates a point where parallel lines in 3-space appear to converge when projected onto a surface such as a plane or, in this case, a sphere.

1.3 Statement of Results

In this paper, we give a complete answer to the above question for all *non-degenerate* SDs and SODs. That is, **for every k , we provide non-degenerate k -oriented SDs and SODs having the minimum possible number of arcs.**

We are also able to do so for *degenerate* k -oriented SDs and SODs of all possible configurations, provided that $k \leq 5$. For $k \geq 6$ we have sporadic results, but classifying and analyzing the numerous degenerate configurations of the poles remains a challenge.

In addition, we also determine the minimum number of *swirls* that a k -oriented SD or SOD may have in all of the aforementioned cases (with one exception; see below).

Our results are summarized in Table 1.

k -orientation	alignment	swirls	SD arcs	SOD arcs
3-oriented	(2, 2, 2)	8	12	12
4-oriented	(2, 2, 2, 3)	6	9	11
4-oriented	(3, 3, 3, 3)	6	9	10
5-oriented	(2, 2, 2, 2, 4)	6	9	11
5-oriented	(2, 3, 3, 3, 3)	5	8	9
5-oriented	(3, 3, 3, 4, 4)	4–5	8	9
5-oriented	(4, 4, 4, 4, 4)	4	8	9
$(k \geq 6)$ -oriented	$(2, 2, \dots, 2, k-1)$	6	9	11
6-oriented	(3, 3, 3, 3, 3, 3)	4	6	8
6-oriented	(3, 3, 3, 4, 4, 4)	4	6	8
6-oriented	(3, 4, 4, 4, 4, 5)	4	6	8
6-oriented	(4, 4, 4, 5, 5, 5)	4	6	8
$(k \geq 6)$ -oriented	$(\geq 5, \geq 5, \dots, \geq 5)$	4	6	8

■ **Table 1** Minimum numbers of swirls and arcs in k -oriented SDs (two-sided arrangements) and SODs (one-sided arrangements), depending on their alignment. The list includes all non-degenerate configurations for every k , as well as all degenerate configurations up to $k = 5$.

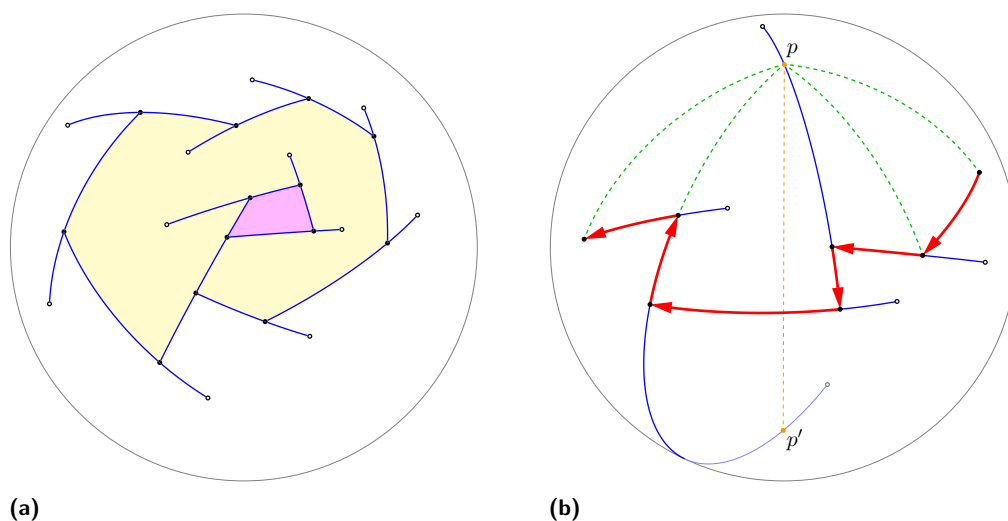
To uniformly describe degenerate and non-degenerate k -oriented SDs with respect to the configuration of their poles, we use the concept of *alignment*. A k -oriented SD with poles p_1, p_2, \dots, p_k has alignment (d_1, d_2, \dots, d_k) if, for every $1 \leq i \leq k$, there are exactly d_i distinct great circles that contain p_i and another pole (we may assume $d_1 \leq d_2 \leq \dots \leq d_k$).

Therefore, non-degenerate k -oriented SDs have alignment $(k-1, k-1, \dots, k-1)$. Another important alignment is $(2, 2, \dots, 2, k-1)$, which corresponds to the visibility map of a polyhedron having only vertical and horizontal edges (Figure 2b), as well as $(3, 3, 3, 3, 3, 3)$, which corresponds to the visibility map of a 4-face-oriented polyhedron (Figure 2c).

Notably, for nearly all of the alignment configurations listed in Table 1, there are SDs and SODs that simultaneously minimize the number of arcs and the number of swirls. The only exceptions are $(3, 3, 3, 4, 4)$, where we do not know if four swirls are possible, and $(4, 4, 4, 4, 4)$, where any SD or SOD minimizing the number of arcs has five swirls (the minimum is four).

Although the last row of Table 1 was partially established in [8, 12, 13], a question was left open: Does any SOD have at least two clockwise and two counterclockwise swirls? In this paper we answer it in the affirmative, not only for SODs, but more generally for all SDs.

The technical proofs of most preliminary results are found in Appendices B–E. Concluding remarks and several open problems are in Appendix F.



■ **Figure 3** (a) The area in yellow is the right-side region of a sliding walk. The eye of a clockwise swirl (in purple) can be found within this region by doing a right-handed sliding walk from its boundary. (b) The initial steps of a clockwise walk with fulcrum p (in red).

2 Basic Constructions and Preliminary Results

2.1 Previous Results

We summarize all previous results on SDs (see also Appendix A). Proofs of the following statements are found in [8, 13]; although they were stated and proved only for SODs, the reader may verify that none of the proofs makes use of the one-sidedness of the arrangements.

Every arc in an SD hits exactly two distinct arcs (one at each endpoint), and no two arcs in an SD intersect in more than one point. The union of all the arcs in an SD is a connected set. An SD with n arcs partitions the unit sphere into $n + 2$ spherically convex regions called *tiles*, and no tile contains antipodal points. The union of the arcs of a swirl separates the unit sphere in two regions, exactly one of which is spherically convex; this region is called the *eye* of the swirl. Given an SD, the interior of any great semicircle on the unit sphere is crossed by at least one arc of the SD, and the interior of any hemisphere contains the eye of at least one swirl. The *swirl graph* of an SD is the undirected multigraph on the set of swirls having an edge between two swirls for every arc shared by them; this graph is planar. Every SD has at least four swirls, including a clockwise one and a counterclockwise one.

In addition to the above properties, SODs also enjoy the following ones. The swirl graph of any SOD is simple, planar, and bipartite. Also, every SOD has at least eight arcs.

2.2 Sliding Walks

Let \mathcal{D} be an SD, and let D be the union of the arcs in \mathcal{D} . Note that any point $p \in D$ lies in the interior of exactly one arc $r(p) \in \mathcal{D}$. A *sliding walk* on \mathcal{D} is defined as a continuous function $w: [0, +\infty) \rightarrow D$ such that, for every maximal interval $I \subseteq [0, +\infty)$ with $|r(w(I))| = 1$, I is left-closed and right-open, and w restricted to I is a unit-speed regular curve.

That is, a sliding walk continues to move in the same direction at unit speed along the same arc $a \in \mathcal{D}$ until it reaches one of its endpoints. Then it turns left or right into the (unique) arc hit by a , follows it until one of its endpoints, and so on indefinitely.

6 Minimal Arrangements of Spherical Geodesics

Since an SD \mathcal{D} has finitely many arcs, a sliding walk w cannot be an injective function, and there exists a minimum x and a unique interval $[x, x'] \subset [0, +\infty)$ such that the restriction of w to $[x, x']$ is a simple closed curve (it is easy to see that either $x = 0$ or $w(x)$ is an endpoint of an arc of \mathcal{D}). This curve separates the unit sphere into a *right-side region* and a *left-side region*. Note that these two regions are well defined and unique for any sliding walk.

A *right-handed walk* (resp., *left-handed walk*) is a sliding walk that always turns right (resp., left) upon hitting a new arc.

► **Observation 5.** *The right-side region (resp., left-side region) of any right-handed walk (resp., left-handed walk) on an SD \mathcal{D} coincides with the eye of a clockwise swirl (resp., counterclockwise swirl) of \mathcal{D} .* ◀

The following lemma is illustrated in Figure 3a; a complete proof is in Appendix B.

► **Lemma 6.** *The right-side region (resp., left-side region) of any sliding walk on an SD \mathcal{D} contains the eye of a clockwise swirl (resp., counterclockwise swirl) of \mathcal{D} .* ◀

If p is a point on the unit sphere, a *clockwise walk* (resp., *counterclockwise walk*) with *fulcrum* p is a sliding walk that moves clockwise (resp., counterclockwise) around p in a weakly monotonic fashion, without touching p or the point p' antipodal to p .² In Figure 3b, the initial steps of a clockwise walk with fulcrum p are shown.

We are ready to settle the open problem stated in [12, Conjecture 2], [13, Conjecture 22], and [8, Section 6]. In fact, we prove it not only for all SODs, but more generally for all SDs.

► **Theorem 7.** *Any SD has at least two clockwise swirls and two counterclockwise swirls.*

Proof. Let \mathcal{D} be an SD; it suffices to prove that \mathcal{D} has two clockwise swirls. Let S be any clockwise swirl of \mathcal{D} (found, for instance, as in Observation 5), and let E be the eye of S . Let p be any point in the interior of E , and let w be a counterclockwise walk with fulcrum p starting from the boundary of E .

Since S is a clockwise swirl, whenever w is in the interior of an arc $a \in S$, it moves toward the endpoint of a not on E . In particular, w never reaches the interior of E , and therefore the left-side region of w contains E . By Lemma 6, the right-side region of w contains the eye of a clockwise swirl, which cannot be E . Hence, \mathcal{D} has a clockwise swirl other than S . ◀

2.3 Swirl Adjacency

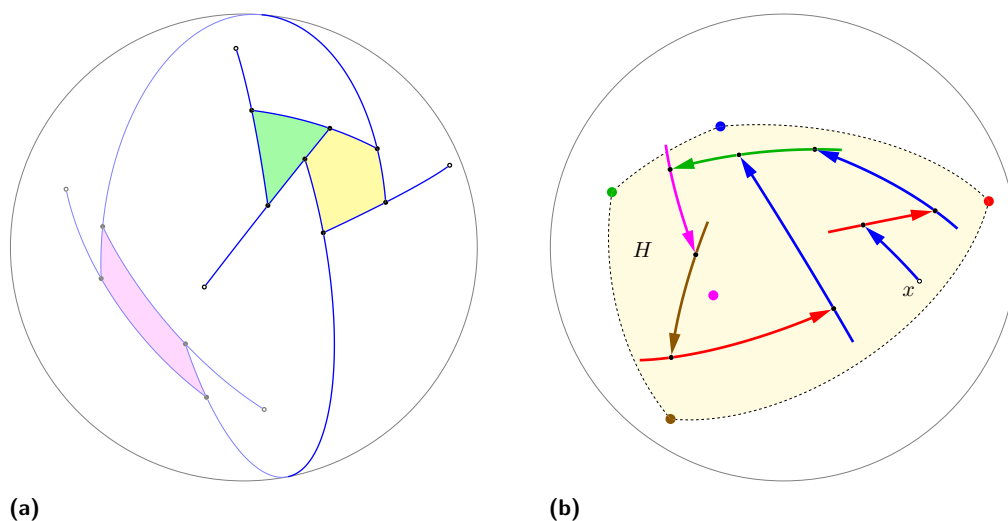
The missing proofs of the results in this section are found in Appendix D.

Recall from Section 2.1 that the swirl graph of any SOD is simple, i.e., any two swirls of an SOD can have at most one arc in common. Moreover, any arc can contribute to at most two swirls. On the contrary, the swirl graph of a general SD may be a non-simple multigraph, since two swirls in an SD may share more than one arc, as shown in Figure 4a. Also, a single arc of an SD can contribute to up to four different swirls (one on each side of each endpoint).

There are essentially two different ways in which two swirls may share multiple arcs. The first configuration involves two *contiguous swirls* whose eyes are adjacent along an arc a of the SD and also share an endpoint of a . An example is given by the yellow and the green swirl in Figure 4a.

► **Proposition 8.** *In any SD, pairs of contiguous swirls share exactly two arcs.* ◀

² In [13], such a walk is called “monotonic walk”, and p is called “pole”. We adopted a different terminology in this paper to avoid confusion with the poles of k -oriented SDs.



■ **Figure 4** (a) The swirl in yellow shares two arcs with the contiguous swirl in green and two arcs with the non-contiguous swirl in purple. (b) An attractor hull and a sliding walk within its interior.

Even if two swirls are not contiguous, they may still share multiple arcs: an example is given by the yellow and purple swirls in Figure 4a.

► **Proposition 9.** *In any SD, a swirl of degree d may share at most $\lfloor d/2 \rfloor$ arcs with the same non-contiguous swirl.* ◀

Two swirls are said to be *discordant* if one of them is clockwise and the other is counter-clockwise; they are *concordant* otherwise.

► **Proposition 10.** *In an SD, let S_1 and S_2 be two swirls that share more than one arc. The following statements are equivalent: (i) S_1 and S_2 are not contiguous; (ii) S_1 and S_2 are concordant; (iii) the eyes of S_1 and S_2 have antipodal interior points.* ◀

2.4 Attractors

The missing proofs of the results in this section are found in Appendix E.

► **Definition 11.** *An attractor of a k -oriented SD is a set of k points, no two of which are antipodal, chosen among its poles and anti-poles. An attractor hull is the spherical convex hull of an attractor.*

Since no two poles are antipodal, a k -oriented SD has exactly 2^k distinct attractors.

► **Observation 12.** *Given any attractor A of a k -oriented SD \mathcal{D} , each arc of \mathcal{D} has a unique vanishing point in A , and is collinear with it.* ◀

► **Proposition 13.** *The k poles of a k -oriented SD cannot be all collinear. Thus, all attractor hulls have non-empty interiors.* ◀

To state the following results, we have to define the terms *intersect*, *cross*, and *thrust*. By *intersect* we simply mean “have a non-empty intersection”. Thus, two arcs intersect even if they only share an endpoint. Instead, an arc a *crosses* a curve γ when their intersection includes an isolated point x internal to a , such that any neighborhood of x contains points

of γ lying on both sides of a . Let R be a spherical polygon contained in the interior of a hemisphere. We say that an arc a *thrusts* the boundary of R at a point x if a intersects the boundary of R at x , as well as the interior of R .

► **Proposition 14.** *If an arc of a k -oriented SD intersects the interior of an attractor hull H , it intersects the boundary of H in at most one point.* ◀

If \mathcal{D} is an SD and R is a region of the unit sphere, we say that two points x and y are *R -connected* (with respect to \mathcal{D}) if there is a path with endpoints x and y that follows the arcs of \mathcal{D} while remaining within R . If such a path is internal to R (except for its endpoints x and y , which may be on the boundary of R), then x and y are *internally R -connected*.

► **Proposition 15.** *If an arc of a k -oriented SD \mathcal{D} intersects the boundary of an attractor hull H at a point x , then there is an arc of \mathcal{D} that crosses the boundary of H at a point y , such that x and y are H -connected (with respect to \mathcal{D}) and lie on a same edge of H . Moreover, if $x \neq y$, then y is not a vertex of H .* ◀

We say that an attractor hull is *total* if it coincides with the entire unit sphere. If an attractor hull is not total, then it is contained in the interior of a hemisphere.

The following result justifies the choice of the term “attractor”.

► **Proposition 16.** *Let H be an attractor hull of a k -oriented SD \mathcal{D} . Then, there exist arcs of \mathcal{D} that intersect the interior of H . Moreover, any point of intersection between an arc of \mathcal{D} and the interior of H is internally H -connected with the vertices of the eye of a swirl of \mathcal{D} .*

Proof. Let A be an attractor of \mathcal{D} , and let H be the spherical convex hull of A . If H is total, there is nothing to prove, because \mathcal{D} is connected and it has at least a swirl (see Section 2.1).

Otherwise, assume for a contradiction that no arc of \mathcal{D} intersects H . Then, H is contained in the interior of a spherically convex tile T (see Section 2.1). If a is any arc of \mathcal{D} bounding T , then T lies on one side of the great circle γ containing a , and therefore γ does not intersect H . Hence, a is not collinear with any point in $A \subset H$, contradicting Observation 12. Consequently, there is an arc of \mathcal{D} that intersects H , and by Proposition 15 there is also an arc that intersects the interior of H (note that H has an interior, due to Proposition 13).

Now, let x be any point of intersection between an arc of \mathcal{D} and the interior of H , and let w be a sliding walk that starts from x and follows each arc in the direction of its vanishing point in A , which exists due to Observation 12. An example of such a sliding walk is shown in Figure 4b. Since H is convex and w always moves toward a point of $A \subset H$ without ever reaching it, we conclude that w never leaves the interior of H . Thus, either the left-side or the right-side region of w is entirely contained in the interior of H . In turn, this region contains the eye of a swirl, due to Lemma 6. ◀

► **Corollary 17.** *In any k -oriented SD \mathcal{D} , the interior of any attractor hull contains the eye of a swirl of \mathcal{D} .* ◀

► **Proposition 18.** *Given an SD \mathcal{D} , any great circle on the unit sphere is crossed by at least three arcs of \mathcal{D} .* ◀

► **Proposition 19.** *Let H be a non-total attractor hull of a k -oriented SD \mathcal{D} , and let x be a point of intersection between an arc of \mathcal{D} and the interior of H . Then, there are three distinct arcs of \mathcal{D} that thrust the boundary of H at three distinct points, not all lying on the same edge of H , all of which are internally H -connected with x .* ◀

The previous result cannot be improved, as there are k -oriented SDs whose arcs thrust the boundary of an attractor hull at exactly three points, two of which lie on the same edge.

3 Minimal Arrangements

In this section, we will derive all the results listed in Table 1.

► **Proposition 20.** *There are no 1-oriented or 2-oriented SDs.*

Proof. The poles of such SDs would be collinear, which contradicts Proposition 13. ◀

An *attractor partition* for a k -oriented SD \mathcal{D} is a collection of internally disjoint non-total attractor hulls of \mathcal{D} that collectively cover the unit sphere. The *vertices* and *edges* of an attractor partition are the vertices and edges of the attractor hulls that constitute it.

► **Lemma 21.** *Let \mathcal{D} be a k -oriented SD with an attractor partition \mathcal{S} consisting of m attractor hulls. Then, \mathcal{D} has at least m non-contiguous swirls, each of which has an eye in the interior of a different attractor hull of \mathcal{S} . Moreover, \mathcal{D} has at least $\lceil 3m/2 \rceil$ arcs that thrust edges of \mathcal{S} , and no arc of \mathcal{D} thrusts the boundaries of more than two attractor hulls.*

Proof. Since the attractor hulls constituting \mathcal{S} are internally disjoint, Proposition 16 guarantees that each of their interiors contains a swirl of \mathcal{D} . Such swirls are distinct and non-contiguous, because their eyes do not share any boundary points.

Moreover, Proposition 19 ensures that at least three arcs of \mathcal{D} thrust the boundary of each attractor hull of \mathcal{S} . Also, no arc of \mathcal{D} thrusts the boundary of any attractor hull of \mathcal{S} at more than one point, due to Proposition 14. Hence, no arc of \mathcal{D} thrusts the boundaries of more than two attractor hulls of \mathcal{S} . We conclude that \mathcal{D} has at least $\lceil 3m/2 \rceil$ arcs that thrust boundaries of attractor hulls of \mathcal{S} . ◀

3.1 3-Oriented SDs

Observe that no 3-oriented SDs are degenerate, otherwise their three poles would be collinear, contradicting Proposition 13. In fact, there is essentially one possible configuration for the poles of a 3-oriented SD, and the set of its $2^3 = 8$ attractor hulls constitutes an attractor partition. We call each attractor hull of a 3-oriented SD an *octant* (see Figure 5a).

► **Theorem 22.** *Any 3-oriented SD has at least eight swirls and 12 arcs. Moreover, there are matching examples that are SODs.*

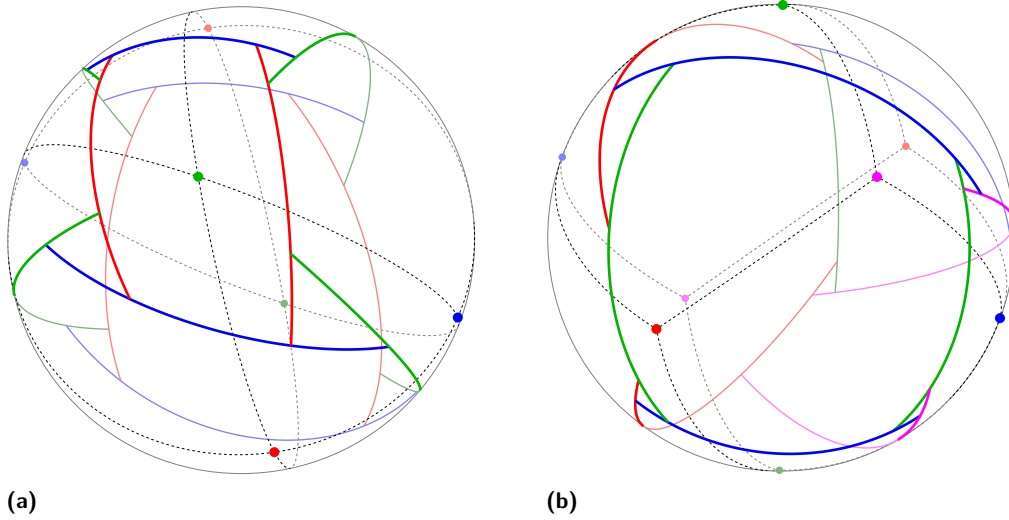
Proof. Since any such SD has an attractor partition consisting of $m = 8$ octants, it follows from Lemma 21 that it has at least $m = 8$ swirls and $3m/2 = 12$ arcs. A 3-oriented SOD with exactly eight swirls and 12 arcs is shown in Figure 5a. ◀

3.2 4-Oriented SDs

There are essentially two possible configurations of four poles on a sphere. Starting from three poles determining eight octants, a fourth pole can be placed either in the interior of an octant, giving rise to a non-degenerate configuration with alignment $(3, 3, 3, 3)$, or on the boundary between two octants, giving rise to a degenerate configuration with alignment $(2, 2, 2, 3)$. Due to Proposition 13, there are no other configurations.

Every non-degenerate 4-oriented SD has an attractor partition consisting of six strictly convex quadrilaterals, which we call *sextants*.³ An example is in Figure 5b.

³ The term “sextant” traditionally refers to a navigation instrument. Its name is derived from the fact that a sextant’s arc covers 60 degrees, which is one sixth of a circle. In this paper, we use the term to indicate each of six regions that partition a sphere.



■ **Figure 5** (a) A 3-oriented SD with exactly eight swirls and 12 arcs. (b) A non-degenerate 4-oriented SOD with exactly six swirls and ten arcs. Each arc has the same color as its vanishing points. Dashed lines mark the boundaries of octants and sextants, respectively.

► **Theorem 23.** *Any non-degenerate 4-oriented SD has at least six swirls and nine arcs; if it is an SOD, it has at least ten arcs. The bounds are tight.*

Proof. Since any such SD has an attractor partition consisting of $m = 6$ sextants, it follows from Lemma 21 that it has at least $m = 6$ non-contiguous swirls and $3m/2 = 9$ arcs. These six swirls, one per sextant, are called *sextant swirls*.

It remains to prove that any non-degenerate 4-oriented SOD \mathcal{D} has at least ten arcs. Let G be the subgraph of the swirl graph of \mathcal{D} induced by its sextant swirls. As we recalled in Section 2.1, the swirl graph of \mathcal{D} is simple, planar, and bipartite; therefore, so is G . We will prove that G has at most eight edges.

Recall that G has six vertices. If the partite sets of G have 1 and 5 vertices, then G has at most five edges. If the partite sets of G have 2 and 4 vertices, then G has at most eight edges. Finally, if the partite sets of G have 3 and 3 vertices, then G has at most eight edges, because the complete bipartite graph $K_{3,3}$ is not planar. Thus, G has at most eight edges.

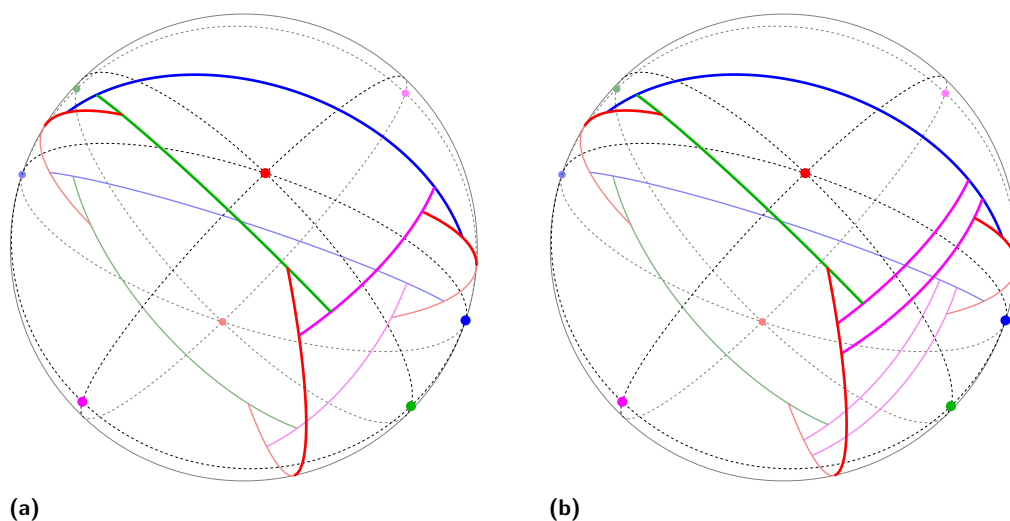
Since a swirl involves at least three arcs, and each arc of an SOD contributes to at most two swirls, the arcs of \mathcal{D} that are involved in sextant swirls are at least $6 \times 3 - 8 = 10$, and hence \mathcal{D} has at least ten arcs.

A non-degenerate 4-oriented SD with exactly six swirls and nine arcs is obtained by perturbing the poles of the SD in Figure 6a. A non-degenerate 4-oriented SOD with exactly six swirls and ten arcs is shown in Figure 5b. ◀

As for degenerate 4-oriented SDs, we treat them as special cases of k -oriented SDs with alignment $(2, 2, \dots, 2, k - 1)$. It turns out that all of these SDs have the same minimum number of swirls and arcs, for any $k \geq 4$. However, since they do not have an attractor partition for $k > 4$, analyzing them will require an ad-hoc technique.

► **Theorem 24.** *If $k \geq 4$, any k -oriented SD with alignment $(2, 2, \dots, 2, k - 1)$ has at least six swirls and nine arcs; if it is an SOD, it has at least 11 arcs. The bounds are tight.*

Proof. All poles lie on a same great circle γ , except for one pole p . Due to Proposition 18, γ is crossed by at least three arcs, which must have p as a vanishing point. Let H and H' be



■ **Figure 6** (a) A 4-oriented SD with alignment $(2, 2, 2, 3)$ having exactly six swirls and nine arcs. (b) A 4-oriented SOD with alignment $(2, 2, 2, 3)$ having exactly six swirls and 11 arcs.

the two hemispheres bounded by γ , with $p \in H$. Let w be a clockwise walk with fulcrum p starting from the interior of H , with the property that the arcs with vanishing point p are followed in the direction of p (as opposed to the anti-pole $-p$). Clearly, w always remains within H , and it takes at least three distinct arcs with vanishing points on γ to do a complete turn around p . Note that such arcs lie entirely in the interior of H . Similarly, there are at least three arcs lying in the interior of H' , which yields nine arcs in total.

There are $2k - 2$ distinct attractor hulls in H . By Proposition 16, each of them contains the eye of a swirl; let E be one of these eyes. Let γ' be a great circle through p that intersects the interior of E . Note that γ' separates H into two parts, each of which contains an attractor hull, and hence the eye of a swirl distinct from E . Thus, there are at least three swirls whose eyes are in H ; for the same reason, there are at least three swirls whose eyes are in H' .

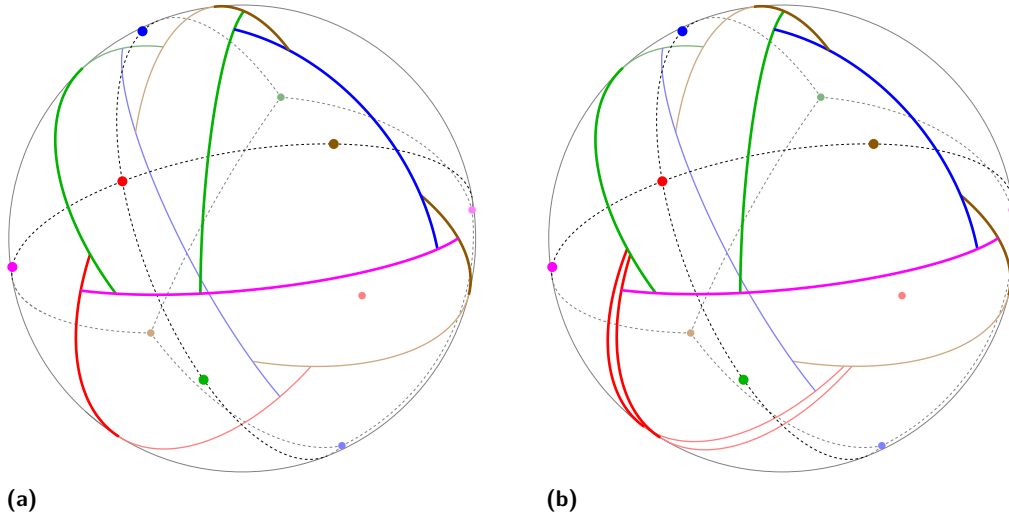
Let G be the subgraph of the swirl graph induced by these six swirls; we will prove that, in an SOD, G has at most seven arcs. Observe that, in any of these swirls, at most two arcs may have p as a vanishing point (otherwise the eye would not be spherically convex), and at most one of them may cross γ . Recall from Section 2.1 that G is simple and bipartite. Hence, at most two arcs may be shared among the three swirls whose eyes are in H , and at most two arcs may be shared among the other three swirls. Thus, G has at most three arcs crossing γ and at most four other arcs. Therefore, the six swirls involve at least $6 \times 3 - 7 = 11$ arcs.

Matching examples for $k = 4$ are shown in Figure 6; adding “dummy poles” on the great circle that already contains three of them yields examples for all $k > 4$, as well. ◀

3.3 5-Oriented SDs

There are essentially four distinct configurations of five poles. By Proposition 13, the poles cannot all be collinear; nonetheless, four of them may lie on a same great circle, giving rise to a configuration with alignment $(2, 2, 2, 2, 4)$, which was already discussed in Theorem 24.

If three poles lie on a same great circle γ , we have two possible alignments: $(2, 3, 3, 3, 3)$ if the two remaining poles are collinear with a pole on γ (see Figure 8a), and $(3, 3, 3, 4, 4)$ otherwise. Each of these alignments completely determines the combinatorics of the configuration.



■ **Figure 7** (a) A 5-oriented SD with alignment $(2, 3, 3, 3, 3)$ having exactly five swirls and eight arcs. (b) A 5-oriented SOD with alignment $(2, 3, 3, 3, 3)$ having exactly five swirls and nine arcs.

Finally, we have the non-degenerate alignment $(4, 4, 4, 4, 4)$. To see why all configurations of five poles with this alignment are equivalent, consider four poles with alignment $(3, 3, 3, 3)$. The great circles through pairs of these poles partition each sextant into four spherical triangles. Since these 24 triangles are all equivalent, placing a fifth pole in the interior of any one of them yields combinatorially equivalent configurations with alignment $(4, 4, 4, 4, 4)$.

► **Theorem 25.** *Any 5-oriented SD with alignment $(2, 3, 3, 3, 3)$ has at least five swirls. Moreover, there are SDs with alignment $(2, 3, 3, 3, 3)$ that have exactly five swirls and eight arcs, and SODs with alignment $(2, 3, 3, 3, 3)$ that have exactly five swirls and nine arcs.*

Proof. As shown in Figure 8a, any SD with alignment $(2, 3, 3, 3, 3)$ has an attractor partition consisting of $m = 5$ attractor hulls. It follows from Lemma 21 that it has at least $m = 5$ non-contiguous swirls. An SD and an SOD with the desired properties are found in Figure 7. ◀

By perturbing the red poles in Figure 7, one obtains configurations of SDs and SODs with alignments $(3, 3, 3, 4, 4)$ and $(4, 4, 4, 4, 4)$ with the same characteristics. In the latter case, however, we can actually construct SDs and SODs with only four swirls (which is minimum).

► **Lemma 26.** *If a 5-oriented SD has exactly four swirls, then it has at least nine arcs. There are matching non-degenerate examples.*

Proof. It is not restrictive to focus on non-degenerate SDs, because the poles of a degenerate SD can be perturbed without altering the number of swirls and arcs, making it non-degenerate.

As argued above, there is essentially a unique non-degenerate configuration of five poles, in the sense that drawing great circles through pairs of such poles yields combinatorially equivalent partitions of the sphere into spherical polygons. Moreover, it is easy to see that there is always a spherically convex pentagonal attractor hull. In fact, without loss of generality, we may assume that the five poles p_1, \dots, p_5 are the vertices of such an attractor hull H ; likewise, the anti-poles p'_1, \dots, p'_5 are also vertices of a spherically convex pentagonal attractor hull H' , as shown in Figure 10.

Assume that there are four swirls; we will identify four interior-disjoint regions of the sphere where their eyes must be located. By Corollary 17, every attractor hull contains

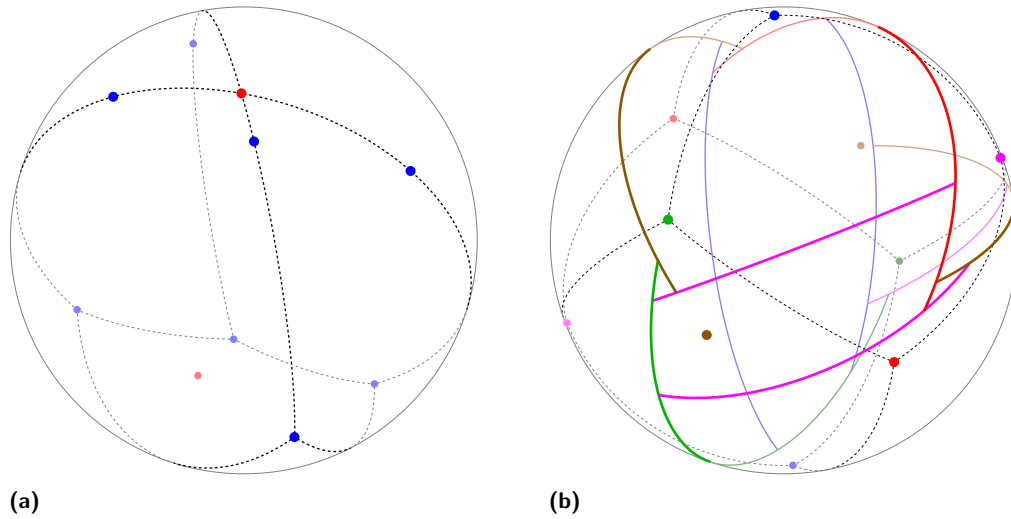


Figure 8 (a) A configuration of five poles with alignment $(2, 3, 3, 3, 3)$ and an attractor partition. (b) A non-degenerate 5-oriented SD with exactly four swirls and nine arcs.

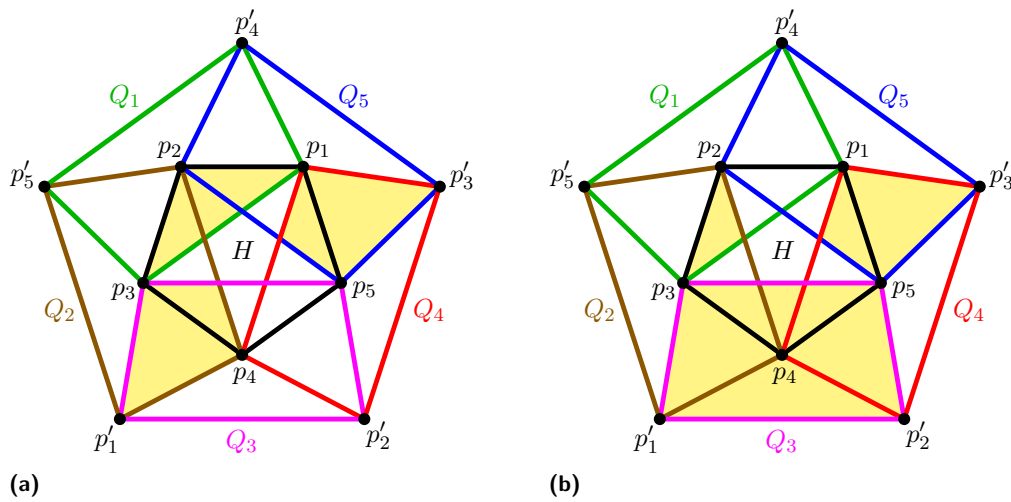
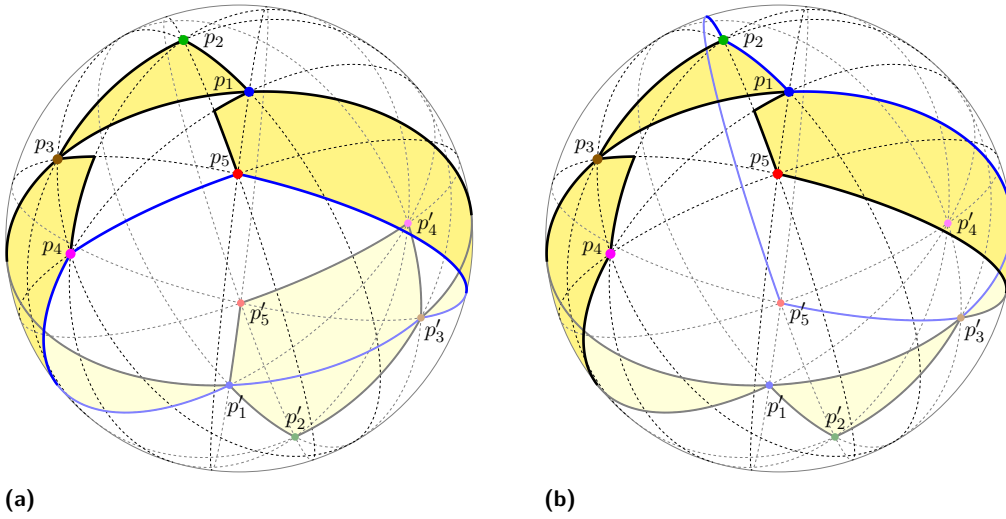


Figure 9 Sketch of the two cases in the proof of Lemma 26.





■ **Figure 10** Case 1 in the proof of Lemma 26.

the eye of a swirl. Hence, H' contains the eye of a swirl: this is the first region. Consider the attractor hull H , as well as the five quadrilateral attractor hulls Q_1, \dots, Q_5 , each of which shares an edge with H' and two non-consecutive vertices with H . For example, Q_1 has vertices p'_4, p'_5, p_3, p_1 and contains p_2 in its interior, etc., as shown in Figure 9. The eyes of the remaining three swirls must be shared among these six attractor hulls.

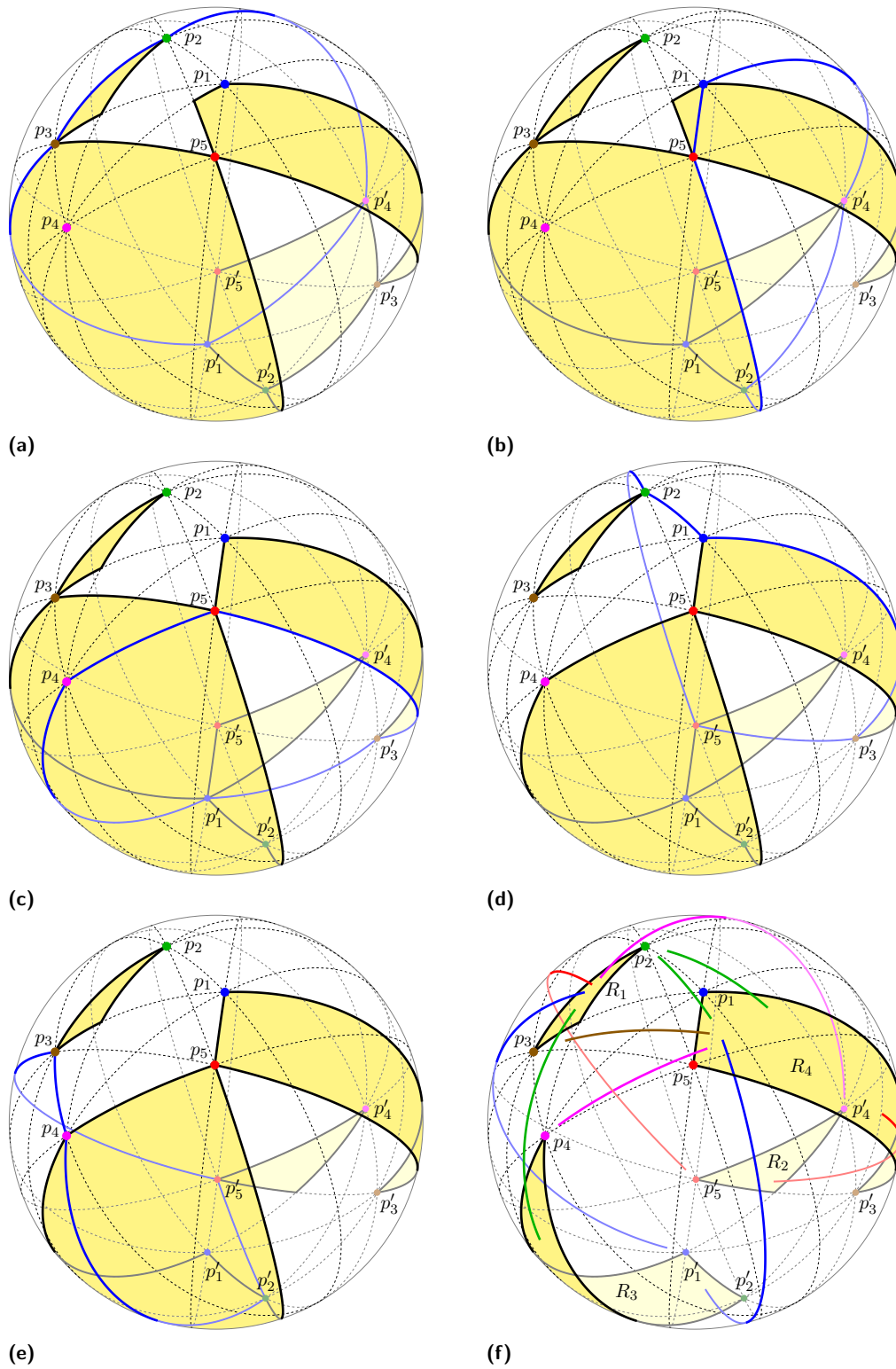
We will distinguish two cases. In Case 1, no eye lies in the intersection of more than two of these six attractor hulls. Since the Q_i 's are all equivalent, we may assume that the three eyes are in the regions $H \cap Q_1, Q_2 \cap Q_3$, and $Q_4 \cap Q_5$, as sketched in Figure 9a. Since the interior of the attractor hull $p'_1 p'_3 p'_5 p_4$ only intersects the chosen region H' (Figure 10a), there must be an eye in their intersection. However, this implies that the attractor hull $p'_3 p'_5 p_2 p_1$ has no eyes in its interior (Figure 10b), which is a contradiction.

In Case 2, an eye lies in the intersection of three of the six attractor hulls, say $H \cap Q_1 \cap Q_2$. Without loss of generality, the other two eyes are in the regions Q_3 and $Q_4 \cap Q_5$, as sketched in Figure 9b. Again, we can progressively reduce these four regions by finding attractor hulls whose interiors intersect only one region, as detailed in Figure 11. Finally, we obtain the four regions R_1, R_2, R_3, R_4 shown in Figure 11f, where the only pairs of regions that may be connected by an arc are $(R_1, R_3), (R_2, R_4)$, and (R_3, R_4) . Indeed, observe that R_2 and R_3 are equivalent to R_1 and R_4 , respectively. Thus, we only have to verify that no arc can connect R_1 with R_2 or with R_4 . Furthermore, due to Proposition 10, no two swirls may share more than one arc, since no two regions have antipodal internal points. Hence, there are at most three arcs that are shared between pairs of swirls. We conclude that the arcs contributing to the four swirls are at least $4 \times 3 - 3 = 9$.

An SD with the desired properties is shown in Figure 8b. ◀

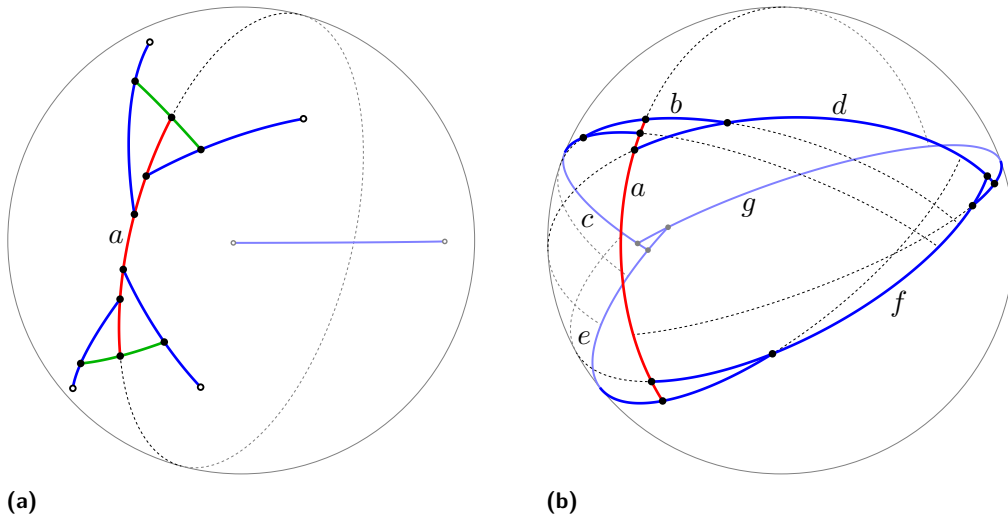
► **Proposition 27.** *In a k -oriented SD, if two non-collinear arcs a and b have a common vanishing point, then the great circle containing a does not intersect b , and vice versa.*

Proof. The vanishing points of a and b are the two intersection points between the great circles containing a and b . By definition of k -oriented SD, b does not intersect its vanishing points, and thus it cannot intersect the great circle containing a , and vice versa. ◀



■ **Figure 11** Case 2 in the proof of Lemma 26.





■ **Figure 12** (a) If an arc a contributes to four swirls, the SD has at least eight arcs. (b) In an SD with seven arcs, if a contributes to three swirls, only b and g may have a common vanishing point.

► **Theorem 28.** *Any 5-oriented SD has at least eight arcs; if it is an SOD, it has at least nine arcs.*

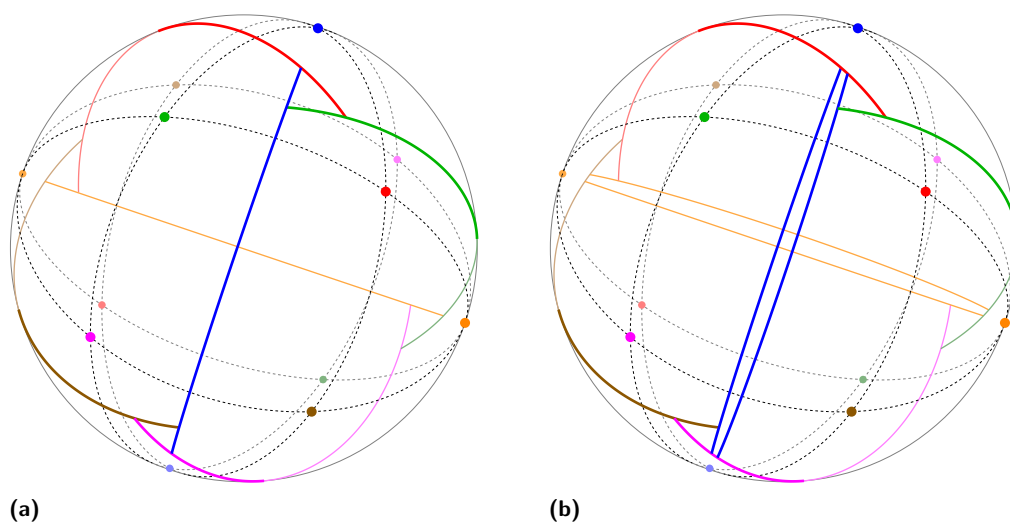
Proof. Due to Theorem 7 and Lemma 26, we may assume that there are at least five swirls.

Recall from Section 2.3 that, since swirls may be contiguous, each arc of an SD may contribute to up to four different swirls. If an arc a contributes to four swirls, it must hit two arcs, and it must block (at least) four arcs, all of which are distinct (see Figure 12a). Of these seven arcs, only the two hit by a cross the great circle containing a . Thus, by Proposition 18, there must be another arc crossing this great circle, implying that there are at least eight arcs in the SD.

Consider now an SD where an arc a is shared by exactly three swirls (two of which are contiguous), and assume that there are fewer than eight arcs. As shown in Figure 12b, two arcs b and e block a (where b contributes to the two contiguous swirls), and three arcs c , d , f hit a . As in the previous case, there must be a seventh arc g crossing the great circle γ containing a , otherwise the only two arcs crossing γ would be b and e . Since there are at least five swirls, the arcs c , d , e , f , g must form two swirls away from a , with eyes located on opposite sides of γ (we recalled in Section 2.1 that the interior of any hemisphere contains the eye of a swirl). We argue that only two of these seven arcs may have a common vanishing point. Indeed, by Proposition 27, incident arcs cannot have a common vanishing point. Also, the extensions of some arcs necessarily hit other arcs, as the dashed lines in Figure 12b illustrate. Again, Proposition 27 implies that such pairs of arcs do not have a common vanishing point. In fact, the only two arcs that may have the same vanishing points are b and g . Hence, there must be at least six distinct poles, and the SD cannot be 5-oriented.

Finally, assume that each arc contributes to at most two swirls, and recall that there are at least five swirls, each of which has degree at least three. Then, the arcs that contribute to such swirls are at least $\lceil (5 \times 3)/2 \rceil = 8$. Consequently, a 5-oriented SD has at least eight arcs.

As for SODs, recall that their swirl graphs are simple and bipartite, and each arc contributes to at most two swirls. Since each of the five swirls has degree at least three, and a bipartite graph on five vertices has at most six edges, we conclude that the arcs contributing to the five swirls are at least $5 \times 3 - 6 = 9$. Thus, a 5-oriented SOD has at least nine arcs. ◀



■ **Figure 13** (a) A 6-oriented SD with alignment $(3, 3, 3, 3, 3, 3)$ having exactly four swirls and six arcs. (b) A 6-oriented SOD with alignment $(3, 3, 3, 3, 3, 3)$ having exactly four swirls and eight arcs.

3.4 6-Oriented SDs and Beyond

As it turns out, 6-oriented SDs and SODs with alignment $(3, 3, 3, 3, 3, 3)$ already achieve the absolute minimum in terms of the number of swirls, as well as the number of arcs.

► **Theorem 29.** *Any SD has at least two clockwise swirls, two counterclockwise swirls, and six arcs. Any SOD has at least eight arcs. Moreover, there are matching examples that are 6-oriented with alignment $(3, 3, 3, 3, 3, 3)$.*

Proof. We recalled in Section 2.1 that any SOD has at least eight arcs; this was proved in [8]. Also, we have proved in Theorem 7 that any SD has at least two clockwise swirls and two counterclockwise swirls. It remains to prove that any SD has at least six arcs.

Let S_1 and S_2 be two clockwise swirls of an SD \mathcal{D} , of degrees $d_1 \geq 3$ and $d_2 \geq 3$, respectively. Without loss of generality, assume that $d_1 \geq d_2$. Due to Proposition 10, S_1 and S_2 cannot be contiguous, because they are concordant.

Thus, by Proposition 9, S_1 and S_2 share at most $\lfloor d_2/2 \rfloor$ arcs. As a consequence, the arcs involved in these two swirls are at most $d_1 + d_2 - \lfloor d_2/2 \rfloor = d_1 + \lceil d_2/2 \rceil$. If $d_1 \geq 4$, then $d_1 + \lceil d_2/2 \rceil \geq 6$, and \mathcal{D} has at least six arcs.

So, assume that $d_1 = d_2 = 3$ and exactly $d_1 + \lceil d_2/2 \rceil = 5$ arcs are involved in S_1 and S_2 . In this case, the two swirls share exactly one arc $a \in \mathcal{D}$. Let γ be the great circle containing a . Of the arcs involved in S_1 and S_2 , two hit a and two block a ; only the latter cross γ . Due to Proposition 18, there must be a third arc that crosses γ , and so \mathcal{D} has at least six arcs.

Matching examples with alignment $(3, 3, 3, 3, 3, 3)$ are found in Figure 13. ◀

By perturbing the poles of the SDs in Figure 13, one obtains a continuum of configurations of 6-oriented SDs and SODs having the absolute minimum number of swirls and arcs. These include degenerate and non-degenerate configurations of all the alignments listed in Table 1 for $k = 6$ (except for $(2, 2, 2, 2, 2, 5)$, which is covered in Theorem 24). Adding “dummy poles” to these configurations yields similar results for $k > 6$. In particular, this covers all non-degenerate configurations for any $k \geq 6$, i.e., those with alignment $(k-1, k-1, \dots, k-1)$.

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A Previous Results

A.1 Properties of SDs

We summarize what is currently known about SDs. Proofs of the following statements are found in [13]; although they were stated and proved only for SODs, the reader may verify that none of the proofs makes use of the one-sidedness of the arrangements.

► **Proposition 30.** *Every arc in an SD hits exactly two distinct arcs, one at each endpoint.* ◀

► **Proposition 31.** *No two arcs in an SD intersect in more than one point.* ◀

► **Proposition 32.** *Given an SD \mathcal{D} , the relative interior of any great semicircle on the unit sphere is crossed by at least one arc of \mathcal{D} .*⁴ ◀

► **Proposition 33.** *An SD partitions the unit sphere into spherically convex regions.* ◀

► **Definition 34.** *A tile is each of the (spherically convex) regions into which the unit sphere is partitioned by an SD.*

► **Corollary 35.** *In an SD, no tile (including its boundary) contains two antipodal points.* ◀

► **Proposition 36.** *Removing any one arc from an SD and taking the union of the remaining arcs yields a connected subset of the unit sphere.* ◀

► **Corollary 37.** *The union of all the arcs in an SD is a connected set.* ◀

► **Proposition 38.** *An SD with n arcs partitions the unit sphere into $n + 2$ tiles.* ◀

► **Proposition 39.** *In an SD \mathcal{D} , let \mathcal{S} be a swirl of degree d .*

- *The union of the d arcs of \mathcal{S} separates the unit sphere in two regions, exactly one of which is spherically convex; this region is a spherical d -gon called the eye E of \mathcal{S} .*
- *The only points of intersection between pairs of arcs of \mathcal{S} are the vertices of E .* ◀

► **Proposition 40.** *In an SD, if the eyes of two distinct swirls have intersecting interiors, then their boundaries are disjoint.* ◀

► **Definition 41.** *The swirl graph of an SD \mathcal{D} is the undirected multigraph on the set of swirls of \mathcal{D} having an edge between two swirls for every arc in \mathcal{D} shared by the two swirls.*

► **Theorem 42.** *The swirl graph of any SD is planar; moreover, any SD has at least one clockwise swirl and at least one counterclockwise swirl.* ◀

The following results on SDs were proved in [8]. Again, they were stated only for SODs, but their proofs do not require the one-sidedness of the arrangements.

► **Lemma 43.** *Given an SD \mathcal{D} , the relative interior of any hemisphere contains the eye of at least one swirl of \mathcal{D} .* ◀

► **Theorem 44.** *Every SD has at least four swirls.* ◀

⁴ In [13], the word “intersect” is used instead of “cross”. However, the proof therein actually yields this slightly stronger result.

A.2 Properties of SODs

In addition to the previous properties, SODs also enjoy the following ones. All proofs are found in [13].

► **Proposition 45.** *In an SOD, any arc coincides with an edge of a tile.* ◀

► **Proposition 46.** *In an SOD \mathcal{D} , let \mathcal{S} be a swirl of degree d , and let E be its eye. Then, the tiles of \mathcal{D} adjacent to E are exactly d ; any two such tiles are either disjoint or intersect only along a single arc of \mathcal{S} .* ◀

► **Theorem 47.** *The swirl graph of any SOD is a simple planar bipartite graph with non-empty partite sets.* ◀

Since there are at least four swirls in an SOD (Theorem 44) and the swirl graph of an SOD is simple, planar and bipartite (Theorem 47), it easily follows that any SOD has at least eight arcs. A proof is found in [8].

► **Corollary 48.** *Every SOD has at least eight arcs.* ◀

B Sliding Walks

We will prove Lemma 6; our technique is inspired by the proof of Lemma 43 (cf. [8, Lemma 9]).

► **Lemma 49.** *The right-side region (resp., left-side region) of any sliding walk on an SD \mathcal{D} contains the eye of a clockwise swirl (resp., counterclockwise swirl) of \mathcal{D} .*

Proof. Let w be any sliding walk and let A be its right-side region. Consider a right-handed walk w' starting from any point on the boundary of A , as shown in Figure 3a. Note that w' never leaves A , because it is a right-handed walk: upon reaching the boundary of A , w' follows it clockwise until it reaches one of its vertices. Then it turns right, either remaining on the boundary of A or entering its interior. In particular, the right-side region E of w' is contained in A . By Observation 5, E is the eye of a clockwise swirl.

For a similar reason, the left-side region of w contains the eye of a counterclockwise swirl. ◀

C Arc Doubling

An *overlap* in an SD is any endpoint shared by two arcs.⁵ Observe that an overlap must lie in the relative interior of some arc.

Let \mathcal{D} be an SD, and let $a \in \mathcal{D}$. For a sufficiently small positive real number ϵ , we define the ϵ -*doubling* of a . This operation consists of replacing a with two disjoint arcs a' and a'' defined as follows. Let x and y be the endpoints of a , located in the relative interiors of arcs $b \in \mathcal{D}$ and $c \in \mathcal{D}$, respectively. Then, a' and a'' also hit b at x' and x'' , respectively, such that the geodesic arc $x'x''$ has length ϵ and its midpoint is x . Similarly, a' and a'' hit c at points ϵ apart with midpoint y . Moreover, any arc of \mathcal{D} that hit a or shared an endpoint with a is now slightly shortened to hit either a' or a'' .

► **Observation 50.** *ϵ -doubling an arc of an SD preserves the number of its swirls.* ◀

⁵ In [13], SODs with overlaps are called “degenerate”. We adopted a different terminology in this paper to avoid confusion with degenerate k -oriented SDs.

► **Proposition 51.** *Any SD, possibly with overlaps, can be converted into an SOD without overlaps by ϵ -doubling of arcs.*

Proof. Observe that ϵ -doubling an arc eliminates any overlaps involving that arc without introducing new overlaps, provided that ϵ is sufficiently small. Furthermore, if an arc is hit from both sides, the ϵ -doubling operation replaces it with two arcs, each of which is hit only from one side. ◀

D Swirl Adjacency

These are the missing proofs from Section 2.3.

► **Proposition 52.** *In any SD, pairs of contiguous swirls share exactly two arcs.*

Proof. Let S_1 and S_2 be contiguous swirls sharing two arcs a and b , with a hitting b at point p . The great circles containing a and b subdivide the unit sphere into four spherical lunes. Each of the eyes of S_1 and S_2 , being spherically convex, lies in one such spherical lune; moreover, the two eyes lie in lunes that are adjacent along a . Thus, if there is a third arc c shared by S_1 and S_2 , then a must hit c at the endpoint opposite to p . However, any arc of a swirl hits exactly one other arc in the same swirl, meaning that c cannot be shared by S_1 and S_2 . ◀

► **Corollary 53.** *If two swirls S_1 and S_2 of an SD \mathcal{D} are contiguous along an arc $a \in \mathcal{D}$, then ϵ -doubling a separates the eyes of S_1 and S_2 and results in a swirl graph where S_1 and S_2 are connected by a single edge.*

Proof. By Proposition 52, the swirls S_1 and S_2 share exactly two arcs a and b . Thus, ϵ -doubling a yields two swirls that share only b . ◀

► **Proposition 54.** *In any SD, non-contiguous swirls may only share arcs that are not consecutive in either swirl.*

Proof. Assume that two swirls S_1 and S_2 share two arcs a and b that are consecutive in S_1 , with a hitting b at a vertex p of the eye of S_1 . By Proposition 39 applied to S_2 , we have that p must be a vertex of the eye of S_2 , as well. It follows that S_1 and S_2 are contiguous. ◀

► **Corollary 55.** *In any SD, a swirl of degree d may share at most $\lfloor d/2 \rfloor$ arcs with the same non-contiguous swirl.*

Proof. In a cycle of d arcs, any subset of more than $\lfloor d/2 \rfloor$ arcs contains a pair of consecutive arcs. If these arcs were shared with the same non-contiguous swirl, Proposition 54 would be contradicted. ◀

► **Proposition 56.** *In an SD, let S_1 and S_2 be two swirls that share more than one arc. The following statements are equivalent.*

- S_1 and S_2 are not contiguous.
- S_1 and S_2 are concordant.
- The eyes of S_1 and S_2 have antipodal interior points.

Proof. Contiguous swirls are obviously discordant. Also, their eyes lie on the same side of one of the two arcs shared by the two swirls. Hence, both eyes lie in the same hemisphere determined by that arc, and therefore cannot have antipodal (interior) points.

We have proved that the first statement is implied by the second and by the third. We will now prove the converse. Let S_1 be a swirl with eye E_1 , and let E'_1 be the spherical polygon antipodal to E_1 . Let a and b be two arcs that are shared between S_1 and a non-contiguous swirl S_2 ; by Proposition 54, a and b are disjoint.

The two great circles containing a and b subdivide the unit sphere into four spherical lunes, one of which, say L_1 , contains E_1 . Assume that the lune containing the eye E_2 of S_2 is adjacent to L_1 . Without loss of generality, E_1 and E_2 lie on the same side of a and on opposite sides of b . However, this implies that S_1 and S_2 are contiguous along b , which is a contradiction.

We conclude that E_2 lies in the lune L_2 opposite to L_1 . Note that a and b , departing from E_1 , follow the edges of L_1 in opposite directions. Then, they continue along the edges of L_2 in the same directions, implying that S_1 and S_2 are concordant swirls.

Let $p_1 \in L_1$ and $p_2 \in L_2$ be the endpoints of a , and let $q_1 \in L_1$ and $q_2 \in L_2$ be the endpoints of b . Since p_1 and q_1 are vertices of E_1 , the segment p_1q_1 is contained in E_1 . Likewise, p_2q_2 is contained in E_2 . Since a and b are shorter than great semicircles, the points p'_2 and q'_2 , antipodal to p_2 and q_2 respectively, lie outside of a and b . Nonetheless, $p'_2, q'_2 \in L_1$. It follows that p_1q_1 and $p'_2q'_2$ cross each other at a point x internal to E_1 . The point x' antipodal to x lies in p_2q_2 , and is therefore internal to E_2 . ◀

E Attractors

These are the missing proofs from Section 2.4.

► **Observation 57.** *Given any attractor A of a k -oriented SD \mathcal{D} , each arc of \mathcal{D} has a unique vanishing point in A , and is collinear with it.* ◀

► **Proposition 58.** *The k poles of a k -oriented SD cannot be all collinear. Thus, all attractor hulls have non-empty interiors.*

Proof. Assume for a contradiction that all poles of a k -oriented SD \mathcal{D} (and therefore all its anti-poles) lie on the same great circle γ . Due to Proposition 32, there is an arc $a \in \mathcal{D}$ that crosses γ , say at x . By definition of k -oriented SD, a is collinear with a pole $f(a) \neq x$. However, since $f(a)$ and x lie on γ , then so does a , contradicting the fact that a crosses γ at x .

Thus, no k points among poles and anti-poles can be collinear, and therefore their spherical convex hull must have a non-empty interior. It follows that all attractors hulls have non-empty interiors. ◀

► **Proposition 59.** *If an arc of a k -oriented SD intersects the interior of an attractor hull H , it intersects the boundary of H in at most one point.*

Proof. By Observation 57, any arc a is collinear with a vanishing point p within H . Also, by definition of k -oriented SD, a does not contain p . It follows that, if a intersects the interior of H , it cannot intersect its boundary in two points, because such points would be on opposite sides of p (or coincident with p). ◀

► **Proposition 60.** *If an arc of a k -oriented SD \mathcal{D} intersects the boundary of an attractor hull H at a point x , then there is an arc of \mathcal{D} that crosses the boundary of H at a point y , such that x and y are H -connected (with respect to \mathcal{D}) and lie on a same edge of H . Moreover, if $x \neq y$, then y is not a vertex of H .*

Proof. If \mathcal{D} has an arc that crosses H at x , there is nothing to prove. Otherwise, Observation 57 implies that \mathcal{D} has an arc a collinear with an edge e of H containing x . Moreover, x lies in the interior of a ; thus, a partially overlaps with e . Hence, there is a vanishing point $v \in e$ that a cannot reach, by Observation 57. So, a hits an arc $a' \in \mathcal{D}$ at a point $y \in e$, strictly between $x \in e$ and $v \in e$. Therefore, y is internal to e , it is H -connected with x , and a' crosses e at y . ◀

► **Proposition 61.** *Let H be an attractor hull of a k -oriented SD \mathcal{D} . Then, there exist arcs of \mathcal{D} that intersect the interior of H . Moreover, any point of intersection between an arc of \mathcal{D} and the interior of H is internally H -connected with the vertices of the eye of a swirl of \mathcal{D} .*

Proof. Let A be an attractor of \mathcal{D} , and let H be the spherical convex hull of A . If H is total, there is nothing to prove, because \mathcal{D} is connected and it has at least a swirl (cf. Corollary 37 and Theorem 7).

Otherwise, assume for a contradiction that no arc of \mathcal{D} intersects H . Then, H is contained in the interior of a spherically convex tile T (cf. Proposition 33). If a is any arc of \mathcal{D} bounding T , then T lies on one side of the great circle γ containing a , and therefore γ does not intersect H . Hence, a is not collinear with any point in $A \subset H$, contradicting Observation 57. Consequently, there is an arc of \mathcal{D} that intersects H , and by Proposition 60 there is also an arc that intersects the interior of H (note that H has an interior, due to Proposition 58).

Now, let x be any point of intersection between an arc of \mathcal{D} and the interior of H , and let w be a sliding walk that starts from x and follows each arc in the direction of its vanishing point in A , which exists due to Observation 57. An example of such a sliding walk is shown in Figure 4b. Since H is convex and w always moves toward a point of $A \subset H$ without ever reaching it, we conclude that w never leaves the interior of H . Thus, either the left-side or the right-side region of w is entirely contained in the interior of H . In turn, this region contains the eye of a swirl, due to Lemma 49. ◀

► **Corollary 62.** *In any k -oriented SD \mathcal{D} , the interior of any attractor hull contains the eye of a swirl of \mathcal{D} .*

Proof. If H is an attractor hull, by Proposition 61 there is a point x of intersection between an arc of \mathcal{D} and the interior of H . Also, there is an eye E of a swirl of \mathcal{D} whose vertices lie in the interior of H and are H -connected with x . Since H is spherically convex, its interior contains E . ◀

It is easy to see that, for a fixed k -oriented SD, any hemisphere contains an attractor hull. Such an attractor hull is unique, provided that no poles lie on the hemisphere's boundary. Thus, we may view Corollary 62 as a stronger version of Lemma 43 for k -oriented SDs.

► **Proposition 63.** *Let R be a spherical polygon contained in the interior of a hemisphere of the unit sphere, and let x be a point of an SD \mathcal{D} that lies in the interior of R . Then, for every great circle γ through x , there are arcs of \mathcal{D} that thrust the boundary of R at two distinct points y and z located on opposite sides of γ (or on γ). Moreover, y and z , are internally R -connected with x .*

Proof. Let p be a point of intersection between γ and the boundary of a hemisphere containing R . Let y (resp., z) be the first point where a clockwise (resp., counterclockwise) walk with fulcrum p starting from x intersects the boundary of R . In the special case where x lies on an arc of \mathcal{D} contained in γ , we choose the two walks to go in different directions, so y and z are necessarily distinct. Also, y and z clearly satisfy the desired conditions. ◀

► **Proposition 64.** *Let R be a spherical polygon contained in the interior of a hemisphere of the unit sphere, and let \mathcal{D} be an SD having an arc with an endpoint x internal to R . Then, there are arcs of \mathcal{D} that thrust the boundary of R in at least three distinct points that are all internally R -connected with x .*

Proof. Let x be an internal point of R where an arc of \mathcal{D} hits another arc. Let γ be any great circle through x ; by Proposition 63, there are two distinct points y and z where arcs of \mathcal{D} thrust the boundary of R . Moreover, x , y , and z are R -connected.

Assume that y and z are not collinear with x . Then there is a great circle γ' through x such that y and z are strictly on the same side of γ' . By Proposition 63, there is a third point w lying on the opposite side of γ' (or on γ') where an arc of \mathcal{D} thrusts the boundary of R . Moreover, x and w are internally R -connected, implying that y , z , and w are internally R -connected.

Assume that y and z are collinear with x . Because x is shared by two arcs $a, b \in \mathcal{D}$, there is a point x' in $R \cap (a \cup b)$ such that x' is not collinear with y and z . Hence, there is a great circle γ' through x' such that y and z are strictly on the same side of γ' (because y and z are not antipodal, since R lies in the interior of a hemisphere). Now we can conclude the proof as in the previous case, observing that x and x' are internally R -connected. ◀

Observe that Propositions 63 and 64 do not require R to be spherically convex or even simply connected.

We can also prove a version of Proposition 64 in the limit case where R is an entire hemisphere.

► **Proposition 65.** *Given an SD \mathcal{D} , any great circle on the unit sphere is crossed by at least three arcs of \mathcal{D} .*

Proof. Let γ be a great circle, and let $p, p' \in \gamma$ be two antipodal points which divide γ into two great semicircles. By Proposition 32, the interior of each of these two semicircles is crossed by an arc of \mathcal{D} . Note that the two crossing points x and y are distinct, because the two semicircles have disjoint interiors. Now, x and y divide γ into two arcs, at least one of which is not shorter than a great semicircle. The interior of this arc is crossed by an arc of \mathcal{D} , again by Proposition 32. Thus, we have found three distinct points where arcs of \mathcal{D} cross γ . Since a geodesic arc cannot cross a great circle in more than one point, there must be three distinct arcs of \mathcal{D} crossing γ . ◀

We can improve Propositions 63 and 64 in the case where R is an attractor hull.

► **Proposition 66.** *Let H be a non-total attractor hull of a k -oriented SD \mathcal{D} , and let x be a point of \mathcal{D} that lies in the interior of H . Then, for every great circle γ through x , there are three distinct arcs of \mathcal{D} that thrust the boundary of H at three distinct points, all internally H -connected with x , two of which lie on strictly opposite sides of γ (i.e., not on γ).*

Proof. Since any non-total attractor hull is a spherical polygon contained in the interior of a hemisphere, Proposition 64 implies that there are at least three points, all internally H -connected with x , where arcs of \mathcal{D} thrust the boundary of H . Moreover, each of these points belongs to a distinct arc of \mathcal{D} , due to Proposition 59. To conclude the proof, it suffices to show that two such points lie on opposite sides of γ .

Proposition 30 implies that x is internal to a unique arc $a \in \mathcal{D}$. By Proposition 59, at least one endpoint x' of a lies in the interior of H ; a hits another arc $a' \in \mathcal{D}$ at x' .

Assume first that a crosses γ at x , i.e., a does not lie on γ . Proceeding as in the proof of Proposition 63, we construct two paths from x , following a in opposite directions away from γ . The points y and z thus obtained are therefore on strictly opposite sides of γ .

Assume now that a does not cross γ , hence it lies on it. Therefore, a' crosses γ at x' , which is internally H -connected with x . Thus, we can repeat the previous argument with x' and a' in lieu of x and a . ◀

► **Proposition 67.** *Let H be a non-total attractor hull of a k -oriented SD \mathcal{D} , and let x be a point of intersection between an arc of \mathcal{D} and the interior of H . Then, there are three distinct arcs of \mathcal{D} that thrust the boundary of H at three distinct points, not all lying on the same edge of H , all of which are internally H -connected with x .*

Proof. By Proposition 66, there are three arcs of \mathcal{D} that thrust the boundary of H at distinct points, all internally H -connected with x . If no edge of H contains all such points, we are finished. Otherwise, all such points lie on the same edge e of H . Also, e is unique, because at least one of the three points is internal to e . Let γ be a great circle through x that does not intersect e (note that γ exists because e is shorter than a great semicircle). By Proposition 66 there is a point, internally H -connected with x and not lying on e , where an arc of \mathcal{D} thrusts the boundary of H . Such an arc is distinct from the previous ones due to Proposition 59. ◀

It is easy to see that the previous results cannot be improved, as there are k -oriented SDs whose arcs thrust the boundary of an attractor hull at exactly three points, two of which lie on the same edge of the attractor hull.

Let A be an attractor of and SD \mathcal{D} , and let H be the attractor hull relative to A , i.e., the spherical convex hull of A . A point of A is a *boundary point* if it lies on the boundary of H , and is an *internal point* if it lies in the interior of H .

► **Proposition 68.** *Let H be a non-total attractor hull relative to an attractor A of an SD \mathcal{D} , and let h_1, h_2, \dots, h_m be the boundary points of A , taken in clockwise order. Let $3 \leq i \leq m$, and let p be the point of intersection between h_1h_{i-1} and h_2h_i . Assume that the interiors of the triangles h_1h_2p and $h_{i-1}h_ip$ are devoid of points of A , while the triangle h_1ph_i contains at most one internal point of A . Then, if a point x of \mathcal{D} lies in the interior of h_1ph_i , there exists an arc of \mathcal{D} that crosses the interior of h_1h_i at a point internally H -connected with x .*

Proof. Let q be the unique internal point of A contained in h_1ph_i , if such point exists. Let w be a sliding walk starting from x with the following properties. As long as w is in the interior of an arc $a \in \mathcal{D}$ with a vanishing point $v \in h_1h_ih_{i+1} \dots h_m$, w proceeds along a in the direction of v . Instead, if a has a vanishing point $v \in ph_2h_3 \dots h_{i-1}$, then w follows a in the direction away from v . If q is a vanishing point of a , then w follows a in the direction of q if and only if w is outside of the triangle h_1qh_i .

Observe that h_1ph_i is the intersection of all the triangles of the form h_1vh_i , with v in the spherical polygon $ph_2h_3 \dots h_{i-1}$. Therefore, w always remains within the interior of h_1ph_i , approaching the triangle h_1qh_i as long as it is outside of it, and then approaching h_1h_i while it is in h_1qh_i . Thus, w eventually crosses the interior of h_1h_i at a point internally H -connected with x . ◀

Note that Proposition 68 cannot be improved, as there are counterexamples where h_1h_2p or $h_{i-1}h_ip$ contains an internal attractor point or h_1ph_i contains two internal attractor points.

► **Corollary 69.** *With the notation of Proposition 68, if a point x of \mathcal{D} lies in the interior of $h_1h_2h_3$, and $h_1h_2h_3$ contains at most one internal point of A , then there is an arc of \mathcal{D} that crosses the interior of h_1h_3 at a point internally H -connected with x .*

Proof. It is immediate from Proposition 68 with $i = 3$. Note that $p = h_2$, and therefore the triangles h_1h_2p and $h_{i-1}h_ip$ degenerate to the edges h_1h_2 and h_2h_3 . Since the interiors of such edges are empty, they are automatically devoid of points of A . ◀

We say that an attractor hull of a k -oriented SD is *void* if it is a spherical k -gon, i.e., there are no poles or anti-poles in its interior. In the case of void attractor hulls, we have an improved version of Proposition 67.

► **Proposition 70.** *Let H be a void non-total attractor hull of a k -oriented SD \mathcal{D} , and let x be a point of \mathcal{D} that lies in the interior of H . Then, there are at least three distinct arcs of \mathcal{D} that cross (respectively, thrust) the boundary of H at distinct points, not all lying on the same two edges of H , all of which are H -connected (respectively, internally H -connected) with x .*

Proof. Note that it suffices to prove that there are arcs of \mathcal{D} that thrust (rather than cross) the boundary of H in at least three distinct points, not all lying on the same two edges of H , all of which are internally H -connected with x . Indeed, given these points, Proposition 60 easily implies that there are also at least three crossing points with the desired properties. In addition, Proposition 59 implies that all such points lie on distinct arcs of \mathcal{D} .

Let Y be the set of points on the boundary of H such that, for all $y \in Y$ there is an arc of \mathcal{D} that thrusts the boundary of H at y , and y is internally H -connected with x . By Proposition 67, Y contains at least three points, and no single edge of H contains all of them. Assume for a contradiction that all points of Y lie within two edges of H , say $e_1 = h_1h_2$ and $e_2 = h_{i-1}h_i$. Without loss of generality, e_1 contains at least two points of Y , say y_1 and y_2 , and e_2 contains at least one point of Y , say y_3 .

Recall that y_1 and y_3 are internally H -connected (with respect to \mathcal{D}) by a path P_1 along \mathcal{D} . Similarly, y_2 and y_3 are internally H -connected (with respect to \mathcal{D}) by a path P_2 . It is easy to recognize that the union of P_1 or P_2 (whose points are all internal to H , except for y_1, y_2, y_3) contains a point x' such that either $x' = p$ or x' lies in the interior of one of the triangles h_1ph_i and $h_2h_{i-1}p$.

We can exclude the case $x' = p$, because any arc $a \in \mathcal{D}$ that has p in its interior and avoids the interiors of h_1ph_i and $h_2h_{i-1}p$ must overlap with h_1h_{i-1} or with h_2h_i . This is because a is collinear with a vertex of H , and there are no vertices of H in the interior of h_1h_2 or in the interior of $h_{i-1}h_i$. Then, by Proposition 59, a has an endpoint in the interior of h_1h_{i-1} or in the interior of h_2h_i , where it hits another arc $a' \in \mathcal{D}$, which in turn intersects the interior of h_1ph_i or $h_2h_{i-1}p$, respectively.

Therefore, without loss of generality, we may assume that x' is in the interior of h_1ph_i . Also, x' is internally H -connected with x , because it lies on P_1 or on P_2 . Note that H and x' satisfy the hypotheses of Proposition 68, because H is void. Thus, there is an arc of \mathcal{D} that crosses h_1h_i at an internal point x'' , which is internally H -connected with x' (hence with x). If h_1h_i is an edge of H , we have reached a contradiction, because $x'' \in Y$, but x'' is not in e_1 or in e_2 .

Otherwise, h_1h_i is a diagonal of H ; let γ be the great circle containing h_1h_i . By Proposition 66, since x'' lies in the interior of H , there is an arc of \mathcal{D} that thrusts the boundary of H at a point z on the side of γ opposite to e_1 and e_2 . Moreover, z is internally H -connected with x'' , and therefore with x . Hence $z \in Y$, which contradicts the fact that z is not in e_1 or in e_2 .

We conclude that no two edges of H contain all points of Y , as desired. ◀

Again, Proposition 70 cannot be improved, in the sense that there are counterexamples where exactly one attractor point lies in the interior of H .

F Open Problems

We conclude this paper with some directions for future research.

Although we have proved all the results listed in Table 1, which include all non-degenerate configurations of k -oriented SDs and SODs, several degenerate configurations are still unexplored. These include 6-oriented SDs with alignment $(2, 3, 3, 3, 4, 4)$ and $(4, 4, 4, 4, 4, 4)$, both of which have two triplets of collinear poles (in the former case, one pole lies at the intersection of the two great circles containing all other poles). In fact, even classifying all degenerate configurations of $k \geq 6$ poles is a challenge in itself.

► **Open Problem 1.** *Extend Table 1 to all degenerate k -oriented SDs and SODs for $k \geq 6$.*

In particular, Table 1 still leaves a small uncertainty on the minimum number of swirls in a 5-oriented SD with alignment $(3, 3, 3, 4, 4)$, which is either four or five. Although the analysis in Lemma 26 can give insights on this configuration, it is not quite sufficient.

► **Open Problem 2.** *Determine if there are 5-oriented SDs with alignment $(3, 3, 3, 4, 4)$ having exactly four swirls.*

Figure 8b shows a non-degenerate 5-oriented SD with exactly four swirls and nine arcs. By ϵ -doubling the green and the red arc, we obtain a non-degenerate 5-oriented SOD with exactly four swirls and 11 arcs. However, there is still a gap between this number and the absolute minimum number of arcs in a non-degenerate 5-oriented SOD, which is nine.

► **Open Problem 3.** *Determine if there exist non-degenerate 5-oriented SODs with exactly four swirls and fewer than 11 arcs.*

In Section 1.1, we remarked that, for any sufficiently large k , there are k -oriented SODs that are not the visibility map of any vertex-hidden point in a polygonal scene, due to the main result of [6]. It would be interesting to determine if this is true for all k .

► **Open Problem 4.** *Determine all values of k such that any k -oriented SOD is the visibility map of a vertex-hidden point in a k -edge-oriented polygonal scene.*

Furthermore, to address our original motivating question in Section 1.2, we would like to find examples of k -edge-oriented polygonal scenes matching the numbers in the last column of Table 1. The main contribution of [8] is a first step in this direction, as it provides a polygonal scene where a vertex-hidden point sees exactly eight edges.

► **Open Problem 5.** *For every k , determine whether there are k -edge-oriented polygonal scenes where a vertex-hidden point sees a number of edges that matches the minimum number of arcs in a non-degenerate k -oriented SOD.*

In Theorem 7, we proved that any SD has at least two clockwise swirls and two counterclockwise swirls. However, this says nothing about whether these swirls are contiguous.

► **Open Problem 6.** *Determine if all SDs have at least two clockwise swirls and two counterclockwise swirls, all of which are non-contiguous.*

Theorem 7 cannot be improved for 4-oriented SDs, since it is possible to slightly modify the SD in Figure 6a to obtain a 4-oriented SD with alignment $(2, 2, 2, 3)$ having exactly two clockwise swirls. Doubling some of its edges also yields an SOD with the same property. However, perhaps Theorem 7 can be improved for 3-oriented SDs.

► **Open Problem 7.** *Determine if all 3-oriented SDs have at least four clockwise swirls and four counterclockwise swirls.*

Since every octant of a 3-oriented SD is a void attractor hull, it immediately follows from Corollary 69 that the interiors of all three edges of each octant must be crossed by arcs. Thus, the swirl graph of any 3-oriented SD with exactly 12 arcs has an induced subgraph isomorphic to the cubical graph. However, this is not enough to characterize the swirl graphs of such minimal 3-oriented SDs: in fact, there are 3-oriented SDs that have more than eight swirls in spite of having exactly 12 arcs (some swirls may revolve around poles and be contiguous to some of the other swirls).

► **Open Problem 8.** *Characterize the swirl graphs of 3-oriented SDs with exactly 12 arcs.*

Observe that, in a k -oriented SD, the degree of a swirl cannot exceed $2k$, because at most two arcs in the swirl may have the same vanishing points. Nonetheless, the degree of a swirl may very well exceed k , although it is shown in [13] that any *swirling* SOD must have a swirl of degree exactly three. Still, not much is known about non-swirling SODs.

► **Open Problem 9.** *Determine if there is a k -oriented SOD where all swirls have degrees exceeding k .*

Theorem 29 implies that there is a continuum of configurations of six poles that allow for the construction of 6-oriented SDs with exactly six arcs. In principle, however, there may be non-degenerate configurations of six (or more) poles that make this construction impossible.

► **Open Problem 10.** *Determine if it is always possible to construct a non-degenerate 6-oriented SD with exactly six arcs, given the locations of its six poles.*