A Theory of Spherical Diagrams

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Abstract

We introduce the axiomatic theory of Spherical Occlusion Diagrams as a tool to study certain combinatorial properties of polyhedra in $\mathbb{R}^3$, which are of central interest in the context of Art Gallery problems for polyhedra and other visibility-related problems in discrete and computational geometry.

1 Introduction

Geometric intuition. Consider a set $\mathcal{P}$ of internally disjoint opaque polygons in $\mathbb{R}^3$ and a viewpoint $v \in \mathbb{R}^3$ such that no vertex of any polygon in $\mathcal{P}$ is visible to v. An example is given by the set of six rectangles in Figure 1 (left) with respect to the point v located at the center of the arrangement.

Let S be a sphere centered at v that does not intersect any of the polygons in $\mathcal{P}$, and let $S_{\mathcal{P}}$ be the set of projections onto S of the portions of edges of polygons in $\mathcal{P}$ that are visible to v (i.e., where polygons occlude projection rays). We call $S_{\mathcal{P}}$ a *Spherical Occlusion Diagram*. Figure 1 (right) shows an example of such a projection.

In this paper we set out to study the combinatorial structure of Spherical Occlusion Diagrams.

Applications. Spherical Occlusion Diagrams naturally arise in visibility-related problems for arrangements of polygons in $\mathbb{R}^3$, and especially for polyhedra.

An example is found in [3], where an upper bound is given on the number of edge guards that solve the Art Gallery problem in a general polyhedron. That is, given a polyhedron $\mathcal{P}$, the problem is to find a (small) set of edges that collectively see the whole interior of $\mathcal{P}$ (refer to [2, 9] for more results on this problem). An edge e sees a point x if and only if there is a point $y \in e$ such that the line segment xy does not properly cross the boundary of $\mathcal{P}$.

The idea of [3] is to preliminarily select a (small) set E of edges that cover all vertices of $\mathcal{P}$. Note that E may be insufficient to guard the interior of $\mathcal{P}$, as some of its points may be invisible to all vertices; Figure 1 (center) shows an example. Thus, an additional (small) set of edges E' is selected, which collectively see all internal points of $\mathcal{P}$ that do not see any vertices. Clearly, $E \cup E'$ is a set of edges that see all internal points of $\mathcal{P}$.

The selection of the edges E' is carried out in [3] by means of an ad-hoc analysis of some properties of points that do not see any vertices of $\mathcal{P}$. Spherical Occlusion Diagrams offer a systematic and general tool to reason about points in a polyhedron that do not see any vertices.

Spherical Occlusion Diagrams have also provided a framework for proving the main result of [8]: Any point that sees no vertex of a polyhedron must see at least 8 of its edges, and that the bound is tight.

2 Axiomatic Theory

Toward an axiomatization. The construction outlined in Section 1 produces an arrangement $S_{\mathcal{P}}$ of arcs on the surface of a sphere S. For each arc $a \in S_{\mathcal{P}}$, let e_a be the edge of a polygon in $\mathcal{P}$ whose orthographic projection on S (partly occluded by other polygons) contains a. Since e_a is a line segment, a must be an arc of a great circle. The fact that each vertex of a polygon in $\mathcal{P}$ is occluded by some other polygon translates into the property that each endpoint of each arc in $S_{\mathcal{P}}$ must lie in the interior of another arc of $S_{\mathcal{P}}$. Also, since e_a is an edge of a polygon $P \in \mathcal{P}$, all arcs of $S_{\mathcal{P}}$ that end on the interior of a must reach it from the same side (as these correspond to edges of polygons partially hidden by P).

Axioms. In the following, S will denote the unit sphere immersed in $\mathbb{R}^3$, and we will abstract from a specific set of polygons $\mathcal{P}$ and a viewpoint v. Some terms will be useful.

Definition 1 *Let a and b be two non-collinear arcs of great circles on a sphere. If an endpoint p of a lies in the relative interior of b, we say that a* hits *b at p (or* feeds into *b at p) and b* blocks *a at p.*

We are now ready to formulate an abstract theory of Spherical Occlusion Diagrams.

Definition 2 *A* Spherical Occlusion Diagram*, or simply* Diagram*, is a finite non-empty collection $\mathcal{D}$ of arcs of great circles on the unit sphere in $\mathbb{R}^3$ satisfying the following axioms.*

A1. If two arcs $a, b \in \mathcal{D}$ have a non-empty intersection, then a hits b or b hits a.

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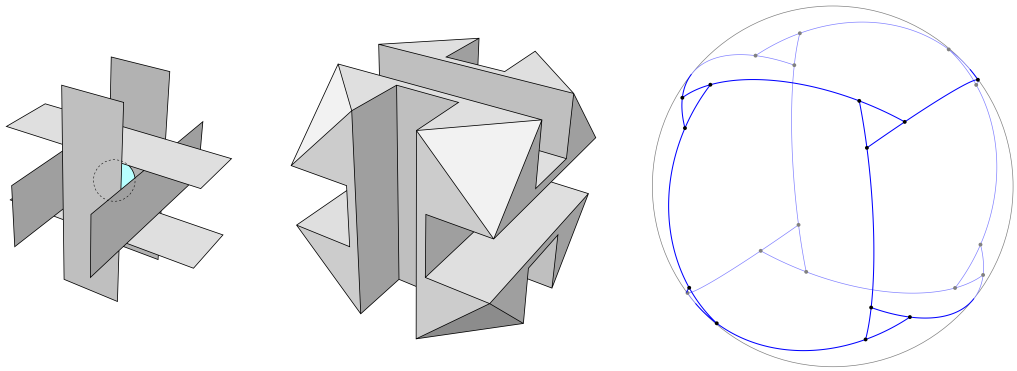

Figure 1: Construction of a Spherical Occlusion Diagram (right) from an arrangement of six rectangles (left) or a polyhedron whose central point does not see any vertices (center)

A2. Each arc in $\mathcal{D}$ is blocked by arcs of $\mathcal{D}$ at each endpoint.

A3. All arcs in $\mathcal{D}$ that hit the same arc of $\mathcal{D}$ reach it from the same side.

Figure 2 shows a Diagram with 18 arcs.

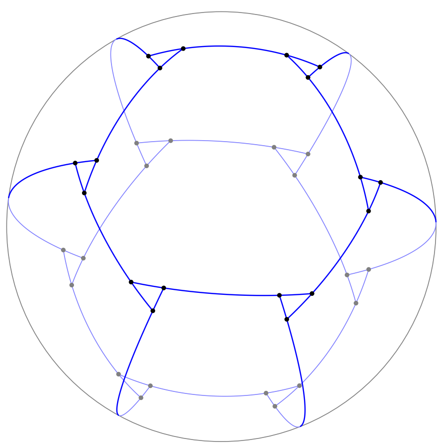

Figure 2: Example of a Diagram with 18 arcs

Realizability. It is immediate to recognize that the Diagrams $S_{\mathcal{P}}$ constructed in Section 1 indeed provide a model for our theory, as they satisfy all its axioms. The proof of the following statement is essentially contained in the first paragraph of Section 2.

Proposition 1 *Any set $S_{\mathcal{P}}$, as constructed in Section 1 for an arrangement of polygons $\mathcal{P}$ and a viewpoint v that sees no vertices of such polygons (re-scaled in such a way that S is the unit sphere), satisfies the axioms of Spherical Occlusion Diagrams, provided that v is in* general position *with respect to $\mathcal{P}$, i.e., no ray emanating from v intersects the interiors of more than two distinct edges of polygons of $\mathcal{P}$.* □

Observe that the general-position requirement in Proposition 1 is irrelevant in the context of the Art Gallery problem and was introduced only for the sake of a more aesthetically pleasing axiomatization of Diagrams. Indeed, the set of points in general position with respect to $\mathcal{P}$ is dense in $\mathbb{R}^3$, whereas the set of points that are visible to any finite set E of edges is topologically closed. Thus, for instance, if the edges in E collectively see all points in general position, then they also see all points that are not in general position.

Although there is compelling evidence that the converse of Proposition 1 is not true, we do not yet have a definitive answer to this fundamental problem, which we leave open. We actually formulate a stronger conjecture. We say that a Diagram $\mathcal{D}$ is *irreducible* if no proper subset of $\mathcal{D}$ is a Diagram.

Conjecture 1 *There is an irreducible Spherical Occlusion Diagram $\mathcal{D}$ (satisfying the conditions in Definition 2) such that $\mathcal{D} \neq S_{\mathcal{P}}$ for any set of internally disjoint polygons $\mathcal{P}$.*

It can be shown that Conjecture 1 is equivalent to its restricted version where $\mathcal{P}$ is a polyhedron of genus zero. Indeed, a set of polygons $\mathcal{P}$ that gives rise to a Diagram $\mathcal{D}$ with respect to a viewpoint v can easily be extended by adding a mesh of polygons whose edges are either shared with $\mathcal{P}$ or concealed from v by polygons in $\mathcal{P}$. The resulting polyhedron gives rise to the same Diagram $\mathcal{D}$.

3 Elementary Properties

We will prove some basic properties of Diagrams.

Proposition 2 *Every arc in a Diagram is strictly shorter than a great semicircle.*

Proof. Referring to Figure 3, assume that an arc a (in red) is at least as long as a great semicircle. Then, taking an endpoint p of a as the North pole and a itself as the prime meridian, consider an arc b_0 that blocks a at p (which exists by Axiom 2). The arc b_0 has exactly one endpoint in the Eastern hemisphere; let b_1 be an arc that blocks b_0 at this endpoint. We can construct a sequence $(b_0, b_1, b_2, \dots)$ of arcs, each of which hits the next at a point of smaller (or equal) latitude, until one of them hits a from the East (this must eventually happen, because a is at least as long as a great semicircle). Note that no b_i other than b_0 can pass through p without contradicting Axiom 1. Symmetrically, we can construct a similar sequence of arcs starting from the endpoint of b_0 that lies in the Western hemisphere. The last arc of this sequence hits a from the West, contradicting Axiom 3. □

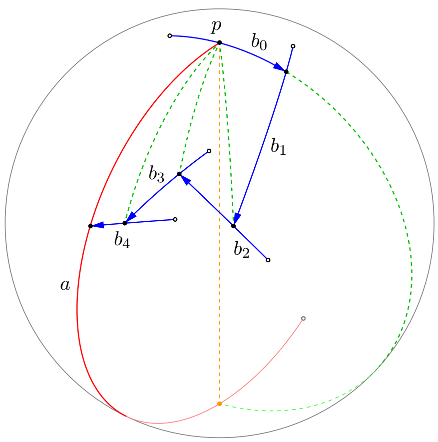

Figure 3: Proof of Proposition 2

We can now prove a stronger form of Axiom 2.

Proposition 3 *Every arc in a Diagram hits exactly two distinct arcs, one at each endpoint.*

Proof. Assume for a contradiction that an endpoint p of an arc a lies in the interior of two arcs b and c. Then b and c intersect at p. By Axiom 1, without loss of generality, b hits c at p, and therefore b and a share an endpoint, which contradicts Axiom 1. Thus, a hits at most one arc at each endpoint; by Axiom 2, it hits exactly one. Moreover, a cannot hit the same arc b at both endpoints p and p', or else p and p' would be antipodal points, and b would be longer than a great semicircle, contradicting Proposition 2. Thus, a hits exactly two distinct arcs. □

Proposition 4 *No two arcs in a Diagram feed into each other.*

Proof. Two arcs feeding into each other must be longer than a great semicircle, as Figure 4 shows. This contradicts Proposition 2. □

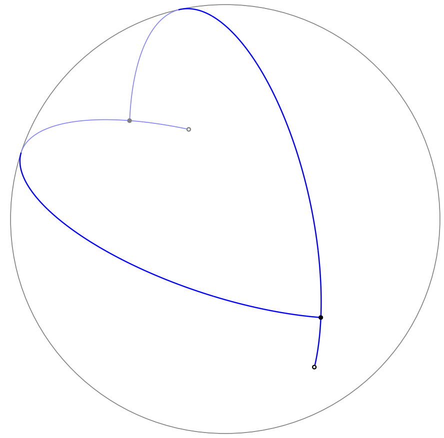

Figure 4: Proof of Proposition 4

Proposition 5 *A Diagram partitions the unit sphere into spherically convex regions.*

Proof. Let $\mathcal{D}$ be a Diagram, and let p and q be two points in the same connected component of $S \setminus \bigcup \mathcal{D}$. There is a chain C of arcs of great circles (drawn in green in Figure 5) that connects p and q without intersecting the Diagram. The arc of a great circle joining p with the third vertex of C (drawn in orange) does not intersect the Diagram either, or else we could reason as in Proposition 2 to construct a sequence of arcs of $\mathcal{D}$ which eventually intersect one of the first two arcs of C. Hence, we can simplify the chain by joining p with its third vertex. Proceeding by induction, we conclude that the arc of a great circle connecting p and q does not intersect $\mathcal{D}$, implying that each connected component of $S \setminus \bigcup \mathcal{D}$ is spherically convex. □

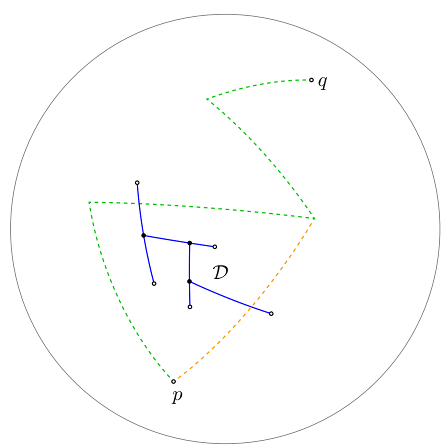

Figure 5: Proof of Proposition 5

Definition 3 *Each of the convex regions into which the unit sphere is partitioned by a Diagram is called a* tile.

It is easy to derive the following from Proposition 5.

Corollary 1 *In a Diagram, the topological closure of no tile contains two antipodal points. Moreover, the relative interior of any great semicircle on the unit sphere intersects some arc of the Diagram.* □

Proposition 6 *The union of all the arcs in a Diagram is a connected set of points.*

Proof. Let the union of the arcs in a Diagram $\mathcal{D}$ have two connected components, given by $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that $\mathcal{D}_1$ and $\mathcal{D}_2$ individually satisfy all axioms, and therefore both are Diagrams. Hence, $\mathcal{D}_2$ is contained in a tile F determined by $\mathcal{D}_1$, as shown in Figure 6. Take two points $p, q \in F$ close to the boundary of F such that the arc of great circle connecting p and q (in orange) intersects $\mathcal{D}_2$. Observe that there exists a chain of arcs of great circle (in green) that connects p and q without intersecting $\mathcal{D}_1$ nor $\mathcal{D}_2$. Hence p and q are in the same tile determined by $\mathcal{D}$. However, since the arc pq intersects $\mathcal{D}$, the tile cannot be spherically convex, contradicting Proposition 5. □

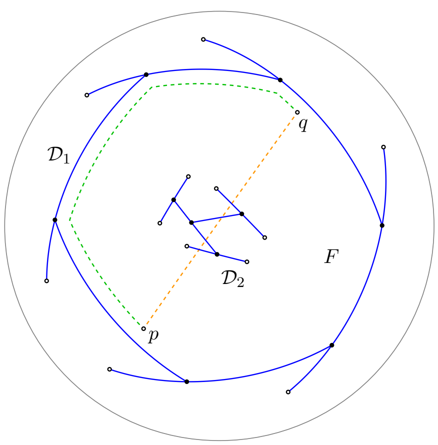

Figure 6: Proof of Proposition 6

Proposition 7 *A Diagram with n arcs partitions the unit sphere into $n+2$ tiles.*

Proof. Every endpoint of an arc of a Diagram divides the arc it hits into two sub-arcs. The set of these sub-arcs induces a spherical drawing of a planar graph with $2n$ vertices and $3n$ edges. Each face of this drawing coincides with a tile of the Diagram. By Euler's formula, the number of faces is $3n - 2n + 2 = n + 2$. □

4 Swirls

There is a curious similarity between Diagrams and continuous vector fields on a sphere. According to the hairy ball theorem, "it is impossible to comb a hairy ball without creating cowlicks". Similarly, it is impossible to construct a Diagram without creating "swirls", as we shall see in this section.

Definition 4 *A* swirl *in a Diagram is a cycle of arcs, each of which feeds into the next going clockwise or counterclockwise. The spherically convex region enclosed by a swirl is called the* eye *of that swirl.*

Figure 2 shows a Diagram with six clockwise swirls and six counterclockwise swirls. Observe that, in an irreducible Diagram, the eye of each swirl coincides with a single tile; in general, the eye of a swirl is a union of tiles, as it may have internal arcs.

Definition 5 *The* swirl graph *of a Diagram $\mathcal{D}$ is the undirected multigraph on the set of swirls of $\mathcal{D}$ having an edge between two swirls for every arc in $\mathcal{D}$ shared by the two swirls.*

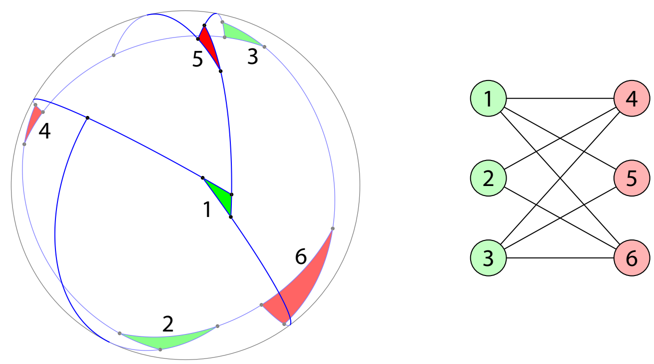

Figure 7: A Diagram and its swirl graph

In Figure 7, the eyes of clockwise swirls are colored green, and the eyes of counterclockwise swirls are colored red. Note that the swirl graph is simple and bipartite; this is true in general.

Theorem 1 *The swirl graph of any Diagram is a simple planar bipartite graph with non-empty partite sets.*

Proof. The swirl graph is spherical, hence planar. It is bipartite, where the partite sets correspond to clockwise and counterclockwise swirls, respectively. Indeed, if the same arc is shared by two concordant swirls (say, clockwise), then it is hit by arcs from both sides, violating Axiom 3.

Figure 8 shows how to find a clockwise and a counterclockwise swirl in any Diagram. For a clockwise swirl, start from any arc and follow it in any direction until it hits another arc. Then turn clockwise and follow this arc until it hits another arc, and so on. The sequence of arcs encountered is eventually periodic, and the period identifies a clockwise swirl. A counterclockwise swirl is found in a similar way.

To prove that the swirl graph is simple, assume for a contradiction that the swirl $\mathcal{S}_1$ shares two arcs a and b with another swirl $\mathcal{S}_2$. Then, the eye of $\mathcal{S}_2$ must be entirely contained in the spherical lune determined by

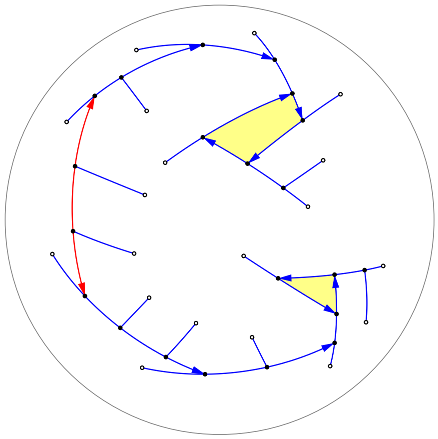

Figure 8: Finding swirls in a Diagram

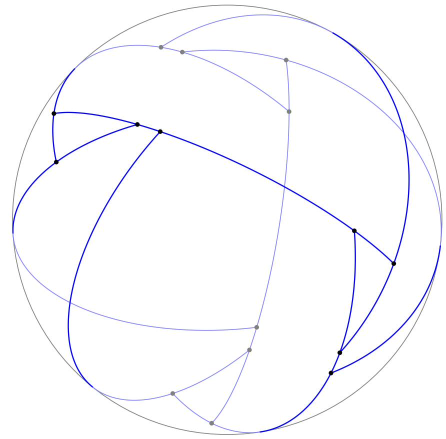

Figure 10: A Diagram with eight arcs and four swirls

a and b, as shown in Figure 9. Since the eye of $\mathcal{S}_2$ is bounded by a, it must lie in the region A. However, the eye of $\mathcal{S}_2$ is also bounded by b, and thus it must lie in the region B. This is a contradiction, since A and B are disjoint. □

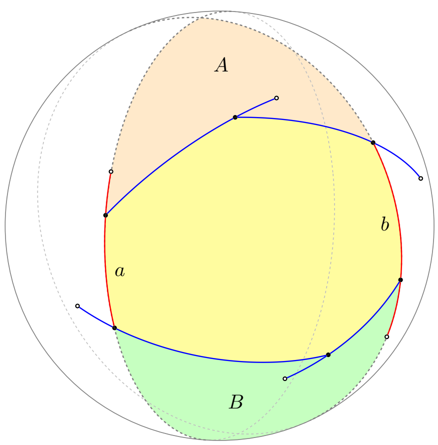

Figure 9: Two swirls cannot share more than one arc

More is actually known about swirl graphs.

Theorem 2 *Every Diagram has at least four swirls.* □

This result has been announced in [8]. From Theorem 2, it easily follows that every Diagram has at least eight arcs. On the other hand, Figure 10 shows an example of a Diagram with exactly eight arcs and exactly four swirls, which is therefore minimal.

It is not yet clear if there are Diagrams with only one clockwise swirl, but we believe this is not the case.

Conjecture 2 *Every Diagram has at least two clockwise and two counterclockwise swirls.*

5 Swirling Diagrams

This section is devoted to a special type of Diagrams whose arcs always meet forming swirls. Patterns arising in these Diagrams are found in modular origami, globe knots, rattan balls, etc.

Definition 6 *A Diagram is* swirling *if every arc is part of two swirls.*

An example of a swirling Diagram is found in Figure 2; further examples are in Figure 14. All of these Diagrams were obtained from convex polyhedra or, equivalently, from convex tilings of the sphere, by a process that we call *swirlification*.

Definition 7 *A subdivision of the unit sphere into strictly convex spherical polygons is* swirlable *if each polygon of the subdivision has an even number of edges.*

Proposition 8 *A subdivision of the unit sphere into strictly convex spherical polygons is swirlable if and only if its 1-skeleton is bipartite.*

Proof. The 1-skeleton is bipartite if and only if its has no odd cycles, which is true if and only if each face has an even number of edges. □

Hence, we can always deform the 1-skeleton of a swirlable tiling, turning each of its vertices into a swirl, going clockwise or counterclockwise according to the bipartition of the 1-skeleton. Conversely, by "shrinking" the eye of each swirl of a swirling Diagram to a point, one obtains a swirlable subdivision of the sphere.

In other words, the swirlification operation establishes a natural correspondence between swirling Diagrams and swirlable subdivisions of the sphere.

Theorem 3 *Every swirling Diagram is the swirlification of a swirlable subdivision of the sphere.* □

Note that we can also obtain a swirlable subdivision of the sphere by taking the dual of a subdivision whose vertices have even degree, or by truncating it. More generally, we have the following.

Proposition 9 *A subdivision of the sphere is swirlable if and only if its truncated dual is swirlable.* □

6 Uniform Diagrams

We now turn to a class of Diagrams that generalizes the swirling ones.

Definition 8 *A Diagram is* uniform *if every arc blocks exactly two arcs.*

Proposition 10 *A Diagram is uniform if and only if every arc blocks at most (respectively, at least) two arcs.*

Proof. By Proposition 3, each arc hits exactly two distinct arcs. Hence, each arc blocks two arcs on average. Thus, if every arc blocks at most two arcs (or at least two arcs), it must block exactly two arcs. □

Proposition 11 *Every uniform Diagram is irreducible.*

Proof. Let $\mathcal{D}$ be a uniform Diagram, and assume that there is a proper subset of arcs $\mathcal{D}' \subset \mathcal{D}$ that is itself a Diagram. By Proposition 6, $\mathcal{D}$ is connected; thus, removing arcs from $\mathcal{D}$ causes some arcs to block fewer than two arcs. Since $\mathcal{D}$ is uniform, it follows that the arcs of $\mathcal{D}'$ block fewer than two arcs on average, contradicting Proposition 3. □

Corollary 2 *In a uniform Diagram, the eye of each swirl coincides with a single tile.*

Proof. If the interior of the eye of a swirl contains some arcs, then such arcs can be removed without violating the Diagram axioms. Hence, such a Diagram is not irreducible, and by Proposition 11 it cannot be uniform. □

Theorem 4 *Every swirling Diagram is uniform.*

Proof. In a swirling Diagram, each arc a is part of two distinct swirls. By Theorem 1, these two swirls share no arcs other than a, and hence a must block one arc from each of them. Therefore, every arc in a swirling Diagram blocks at least two arcs, and by Proposition 10 the Diagram is uniform. □

The converse of Theorem 4 is not true in general, as Figure 11 shows.

Definition 9 *An endpoint of an arc of a Diagram is called a* non-swirling vertex *if it is not incident to the eye of any swirl. A walk on a Diagram is* non-swirling *if it only touches non-swirling vertices and, whenever it touches an arc, it follows it until it reaches one of its endpoints, without touching any other arc along the way. A cyclic non-swirling walk is called a* non-swirling cycle.

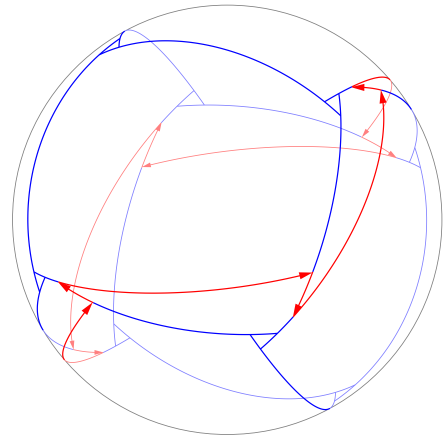

Figure 11: A uniform Diagram that is not swirling

Observe that there is a non-swirling cycle that covers all the non-swirling vertices of the Diagram in Figure 11 (drawn in red). This is not a coincidence.

Theorem 5 *In any uniform Diagram, all non-swirling vertices are covered by disjoint non-swirling cycles.*

Proof. Consider a non-swirling walk W on a uniform Diagram terminating at a non-swirling vertex p_i, endpoint of an arc a_i, as Figure 12 illustrates. We will prove that W can be extended to a longer non-swirling walk in a unique way.

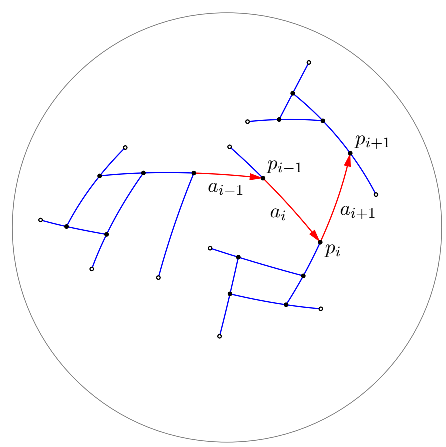

Figure 12: Proof of Theorem 5

Let a_{i+1} be the arc that blocks a_i at p_i. Since exactly two arcs feed into a_{i+1}, there is exactly one endpoint of a_{i+1}, say p_{i+1}, that can be reached from p_i without touching any arc other than a_{i+1}.

By definition of non-swirling walk, p_{i+1} can be used to extend W if and only if it is a non-swirling vertex. However, if p_{i+1} were incident to a swirl's eye E, then an arc of that swirl would either hit a_{i+1} between p_i and p_{i+1}, contradicting the fact that a_{i+1} blocks exactly two arcs, or it would hit a_{i+1} on the other side of p_i, implying that E contains the arc a_i in its interior, which contradicts Corollary 2.

Hence, W can be extended uniquely to a non-swirling walk. By a similar reasoning, we argue that W can also be uniquely extended backwards to a non-swirling walk. Thus, W is part of a unique non-swirling cycle. Now we conclude the proof by inductively repeating the same argument with any remaining non-swirling vertices. □

We can construct uniform Diagrams with any number of arbitrarily long non-swirling cycles. An example with two non-swirling cycles is shown in Figure 13.

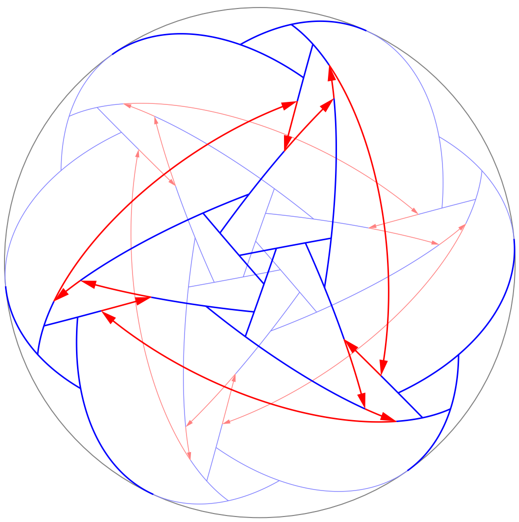

Figure 13: A uniform Diagram with two non-swirling cycles

7 Conclusions

We introduced the theory of Spherical Occlusion Diagrams and studied their basic properties, while also discussing some applications to visibility-related problems in discrete and computational geometry.

Although we strongly believe Conjecture 1 to be true, a related and more subtle question can be asked, inspired by previous work on weaving patterns [1, 7]. Namely, whether for every Diagram $\mathcal{D}$ there is a *combinatorially equivalent* Diagram $\mathcal{D}'$ and a set $\mathcal{P}$ of internally disjoint polygons such that $\mathcal{D}' = S_{\mathcal{P}}$. In other words, does every class of combinatorially equivalent Diagrams contain a realizable instance?

We have introduced three remarkable families of Diagrams: irreducible, uniform, and swirling. We proved that all swirling Diagrams are uniform, and all uniform Diagrams are irreducible; moreover, Theorem 5 reveals a deeper structural connection between swirling and uniform Diagrams. A complementary observation is that it seems to be possible to systematically transform any uniform Diagram into a swirling Diagram by "sliding" arcs' endpoints along other arcs and "merging" coincident arcs. Making this observation rigorous is left as a direction for future work.

More generally, we may wonder which Diagrams can be transformed into swirling ones by sequences of elementary operations on arcs (defining suitable "elementary operations" is in itself an open problem). The Diagram in Figure 10 shows that the question is not trivial. Indeed, this is the unique configuration of any Diagram with eight or fewer arcs; since the Diagram itself is not swirling, it cannot be transformed into a swirling one by means of operations that only rearrange or merge arcs.

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References

[1] S. Basu, R. Dhandapani, and R. Pollack. On the Realizable Weaving Patterns of Polynomial Curves in $\mathbb{R}^3$. In *Proceedings of the 12th International Symposium on Graph Drawing (GD 2004)*, pp. 36–42, 2004.

[2] N. M. Benbernou, E. D. Demaine, M. L. Demaine, A. Kurdia, J. O'Rourke, G. T. Toussaint, J. Urrutia, and G. Viglietta. Edge-Guarding Orthogonal Polyhedra. In *Proceedings of the 23rd Canadian Conference on Computational Geometry (CCCG)*, pp. 461–466, 2011.

[3] J. Cano, C. D. Tóth, J. Urrutia, and G. Viglietta. Edge Guards for Polyhedra in Three-Space. *Computational Geometry: Theory and Applications*, vol. 104, art. 101859, 2022.

[4] D. Eppstein, E. Mumford, B. Speckmann, and K. Verbeek. Area-Universal and Constrained Rectangular Layouts. *SIAM Journal on Computing*, vol. 41, no. 3, pp. 537–564, 2012.

[5] A. I. Merino, T. Mütze. Efficient Generation of Rectangulations via Permutation Languages. In *37th International Symposium on Computational Geometry (SoCG 2021)*, pp. 54:1–54:18, 2021.

[6] J. O'Rourke. Visibility. In J. E. Goodman, J. O'Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, pp. 875–896, CRC Press, 2017.

[7] J. Pach, R. Pollack, and E. Welzl. Weaving Patterns of Lines and Line Segments in Space. *Algorithmica*, vol. 9, no. 6, pp. 561–571, 1993.

[8] C. D. Tóth, J. Urrutia, and G. Viglietta. Minimizing Visible Edges in Polyhedra. In *Proceedings of the 23rd Thailand-Japan Conference on Discrete and Computational Geometry, Graphs, and Games (TJCDCGGG 2020+1)*, pp. 70–71, 2021.

[9] G. Viglietta. Optimally Guarding 2-Reflex Orthogonal Polyhedra by Reflex Edge Guards. *Computational Geometry: Theory and Applications*, vol. 86, art. 101589, 2020.

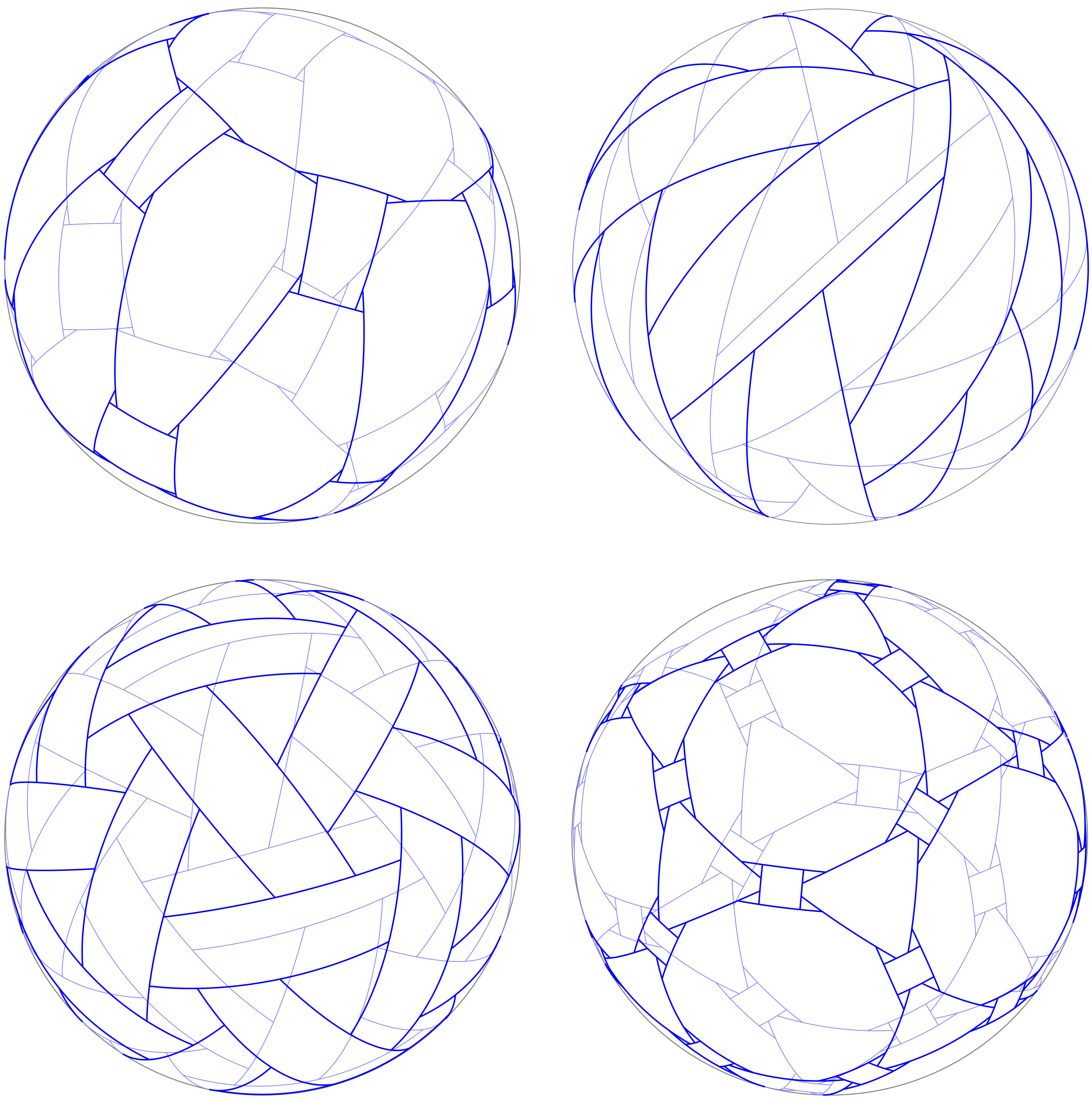

Figure 14: Examples of the swirlification method developed in Section 5 to produce swirling Diagrams from convex polyhedra with a bipartite 1-skeleton (or, equivalently, from swirlable subdivisions of the unit sphere). The pictures show swirling Diagrams resulting from a truncated antiprism, a trapezohedron, a rhombic triacontahedron, and a truncated icosidodecahedron, respectively.