Extended Abstract
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Minimizing Visible Edges in Polyhedra

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Abstract

We prove that, given a polyhedron $\mathcal{P}$ in $\mathbb{R}^3$, every point in $\mathbb{R}^3$ that does not see any vertex of $\mathcal{P}$ must see eight or more edges of $\mathcal{P}$; this bound is tight.

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1 Introduction

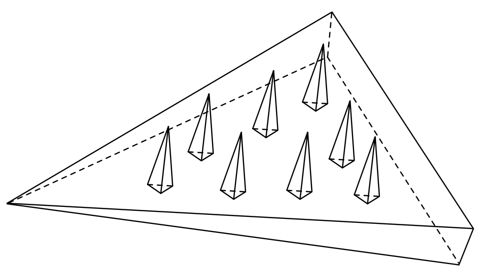
A pivotal result in the study of art gallery or illumination problems is that every simple polygon with n vertices can be illuminated with at most $\lfloor \frac{n}{3} \rfloor$ point lights placed at its vertices. If a simple polygon is orthogonal (i.e., its edges are parallel to the x- or y-axis), $\lfloor \frac{n}{4} \rfloor$ vertices suffice; see [3, 4].

Not much is known about art gallery problems in $\mathbb{R}^3$. We say that a point p in a polyhedron $\mathcal{P}$ *sees* a point q if the line segment pq is completely contained in $\mathcal{P}$. It is known that the vertices of a polyhedron $\mathcal{P}$ do not necessarily see all points in the interior of $\mathcal{P}$ (see Figure 2a for an example). Upper and lower bounds on the number of edges needed to illuminate a polyhedron in $\mathbb{R}^3$ are given in [1, 2, 5].

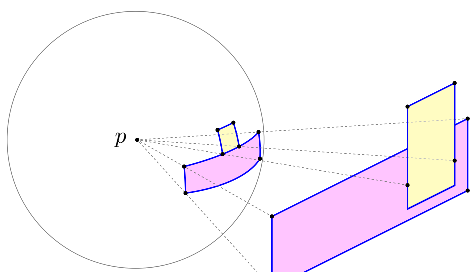
In this paper we show the following:

Theorem 1.1. *Let $\mathcal{P}$ be a polyhedron in $\mathbb{R}^3$. Any point $p \in \mathbb{R}^3$ that sees no vertices of $\mathcal{P}$ sees at least eight of its edges. Moreover, any point $p \in \mathbb{R}^3$ sees at least six edges of $\mathcal{P}$. These bounds are tight.*

In the *brush* polyhedron depicted in Figure 1a, any interior point close to the tip of a tetrahedral "spike" sees exactly six edges. It is not hard to prove that any point $p \in \mathbb{R}^3$ sees at least six edges of *any* polyhedron. We remark that this also holds if p is in the exterior of the polyhedron. Indeed, for an edge to be visible to p, it is sufficient that p sees one of its endpoints.



(a) A brush



(b) Constructing a Spherical Occlusion Diagram

Figure 1: Visibility in $\mathbb{R}^3$

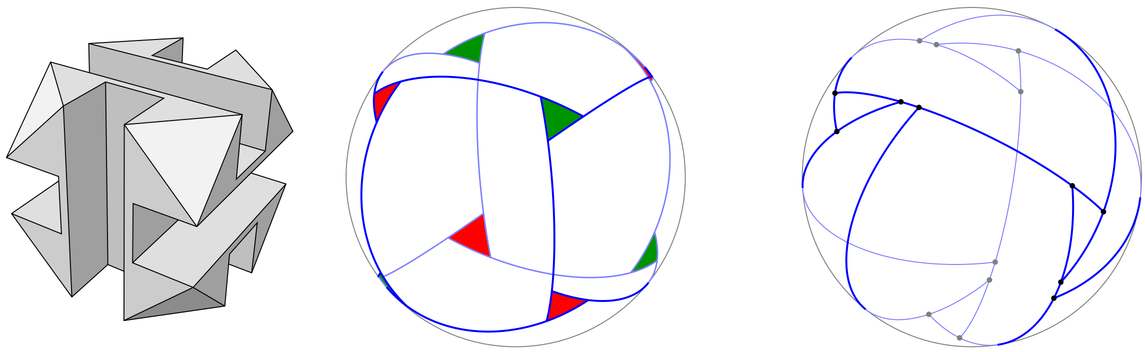
When $\mathcal{P}$ is an orthogonal polyhedron, i.e., when all its faces have normal vectors parallel to one of the coordinate axes, any point in the interior of $\mathcal{P}$ sees at least twelve edges (while any exterior point sees at least eight edges). In fact, take the three planes through an internal point p with normal vectors parallel to the coordinate axes; each of these planes intersects at least four edges visible to p. The bound is tight; it is achieved, for example, in a cube or in the polyhedron in Figure 2a.

2 Spherical Occlusion Diagrams

Proving that, for an arbitrary polyhedron in $\mathbb{R}^3$, any point that sees no vertices can always sees at least eight edges is tricky. To prove our results we study *Spherical Occlusion Diagrams* (SOD), introduced in [6]. SODs arise from the visibility region of a point p that does not see any vertex of a polyhedron $\mathcal{P}$. Specifically, a SOD is the geometric figure obtained by orthographically projecting all visible edge sub-segments onto a small sphere centered at p; see Figure 1b. What is obtained is a spherical (hence planar) arrangement of noncrossing arcs of great circle, and the contact graph of these arcs is a simple planar directed graph where each vertex has outdegree 2. Figure 2a shows the SOD corresponding to the point p at the center of the polyhedron on the left.

Hence, a SOD is a set of noncrossing arcs of great circle on a sphere satisfying the following properties, which are called *diagram axioms* in [6]: (a) if two arcs intersect, one "feeds into" the other; (b) each arc feeds into two arcs; (c) all arcs that feed into the same arc reach it from the same side. These axioms easily imply some properties of SODs:

Theorem 2.1. *The following statements hold for every SOD. (1) No two arcs in a SOD feed into each other. (2) Each arc in a SOD is shorter than a great semicircle. (3) Every SOD is connected. (4) A SOD with n arcs partitions the sphere into $n+2$ spherically convex regions.*



(a) The SOD corresponding to the point p at the center of the polyhedron on the left; the colors of the swirls indicate the bipartition of the swirl graph

(b) A SOD with eight arcs and four swirls

Figure 2: Some Spherical Occlusion Diagrams

A crucial structure in a SOD is a *swirl*, defined as a cycle of arcs such that each arc feeds into the next going clockwise or counterclockwise. In the SOD in Figure 2a, the clockwise swirls are colored green, and the counterclockwise ones are red. The *swirl graph* of a SOD is the undirected multigraph on the set of swirls such that, for each arc shared by two swirls, there is an edge in the swirl graph.

Theorem 2.2. *The swirl graph of a SOD is a simple planar bipartite graph with nonempty partite sets; moreover, every SOD has at least four swirls.*

As a consequence we have the following corollary, which in turn implies our main result, Theorem 1.1:

Corollary 2.3. *Any SOD has at least eight arcs.*

Figure 2b shows an example of a SOD with exactly eight arcs and four swirls.

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