The Bellows Theorem (Introduction)

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Real-life bellows

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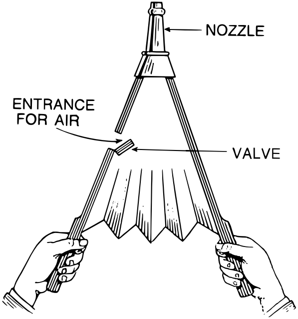
Observation: All of them have elasticity or curved creases. Why?

Definition of bellows

Wiktionary:

"A bellows is a container which is <u>deformable</u> in such a way as to <u>alter its volume</u>, which has an outlet where one wishes to blow air."



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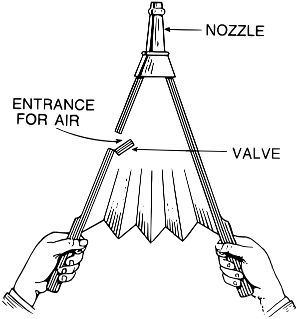
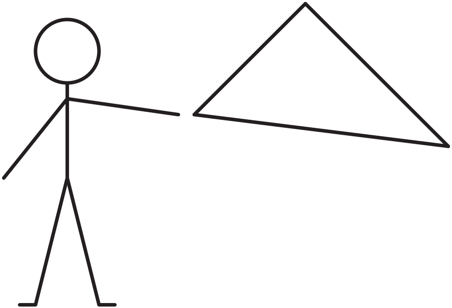
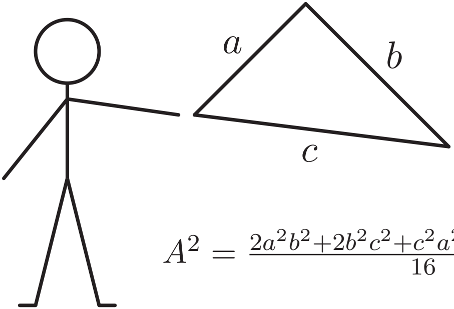


Problem: Is it possible to construct a "geometric bellows" in some mathematical sense?

Bellows in Flatland



A triangular linkage (rigid bars and joints) cannot be a bellows.



$$A^2 = \frac{2a^2b^2+2b^2c^2+c^2a^2-a^4-b^4-c^4}{16}$$

Heron's formula gives its area as a function of the edge lengths.

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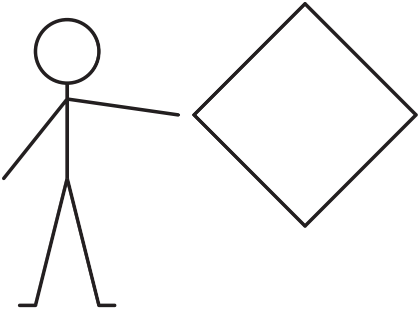

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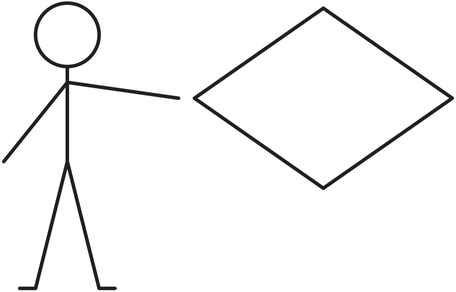

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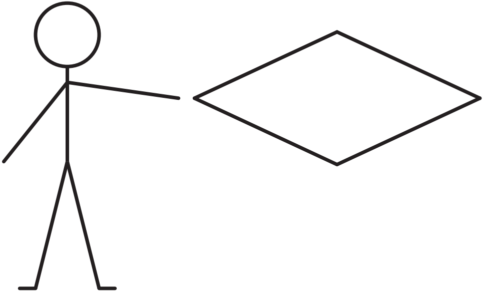

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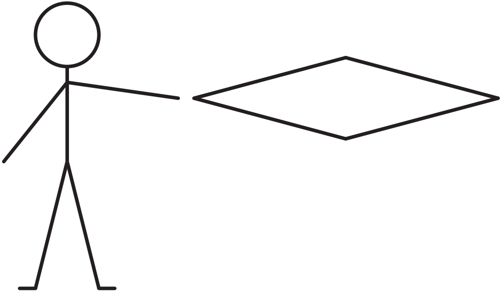

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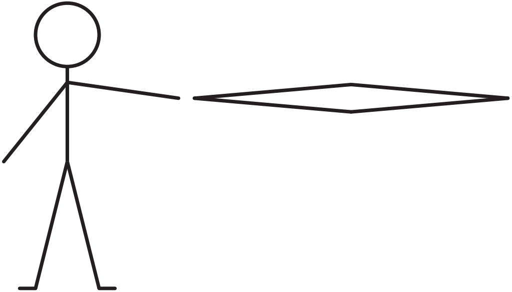

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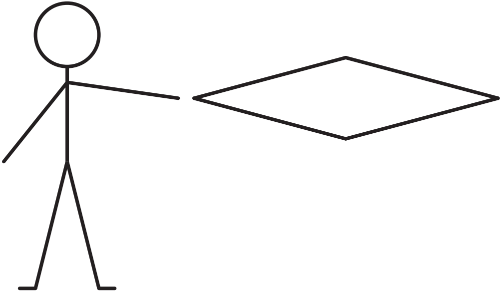

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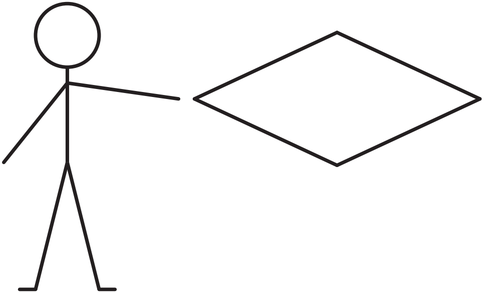

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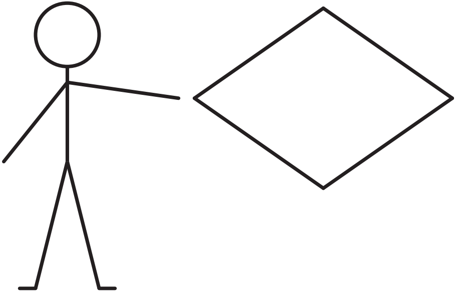

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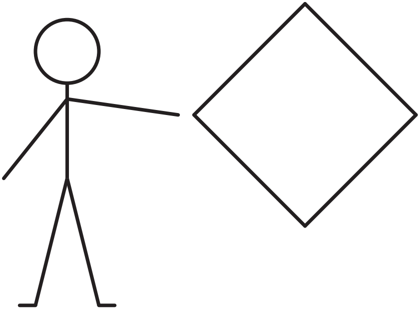

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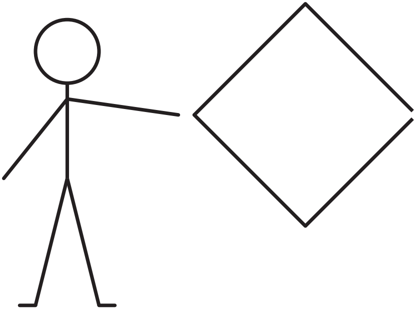

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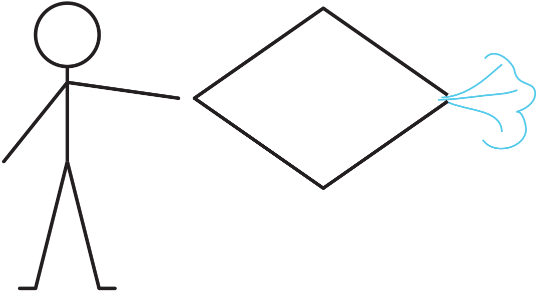

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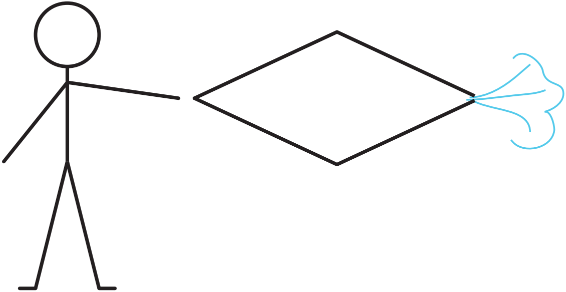

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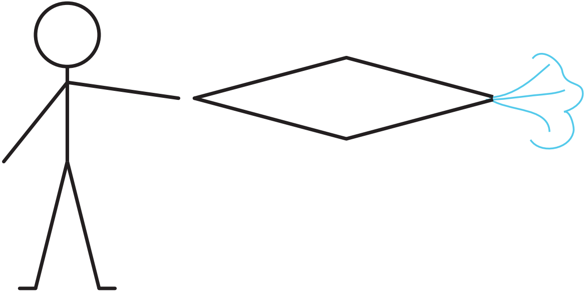

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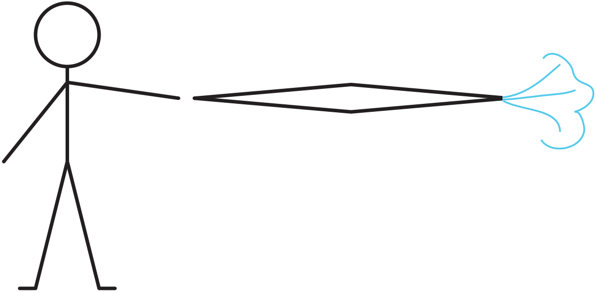

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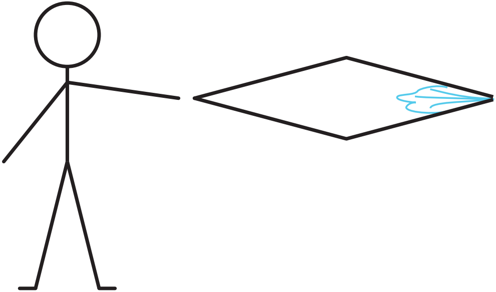

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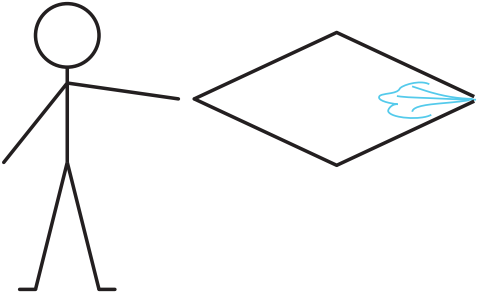

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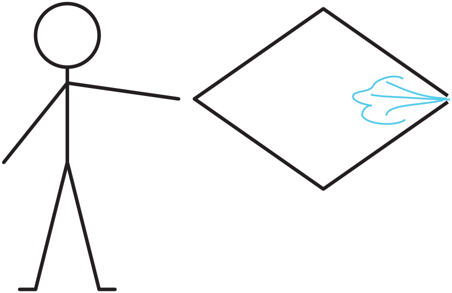

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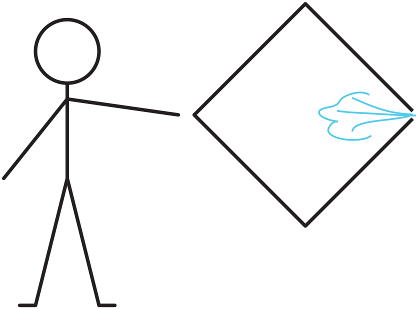

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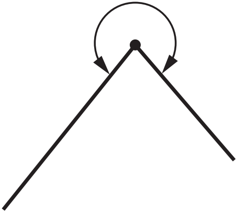
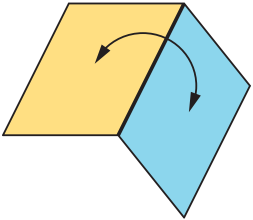
Polyhedral model

To define a 3D model, let us generalize 2D linkages.





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Instead of joints at vertices, we have hinges at edges.

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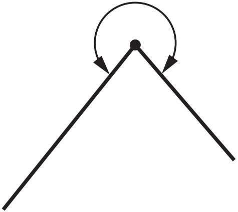

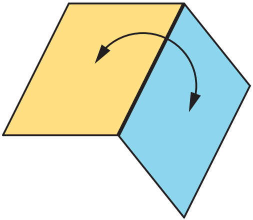

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Instead of joints at vertices, we have hinges at edges.

Problem: Can such a polyhedron be a bellows? Can it even flex?

Euclid's "definition" of equal polyhedra

Euclid's Elements, Book XI, Definition 10:

ι′. Ἴσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

Literal translation:

Equal and similar solid figures are those contained by similar planes equal in multitude and in magnitude.

Modern interpretation:

Two polyhedra are equal if they have the same combinatorial structure and equal corresponding faces.

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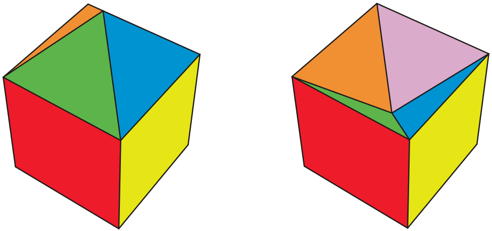
Two polyhedra are equal if they have the same combinatorial structure and equal corresponding faces.

$\Longrightarrow$ Euclid seems to disallow the existence of flexible polyhedra!

Simson's critique to Euclid

Simson, 1756: Euclid's statement cannot be a definition, but a theorem that ought to be proved.

Also, the statement is not universally true:



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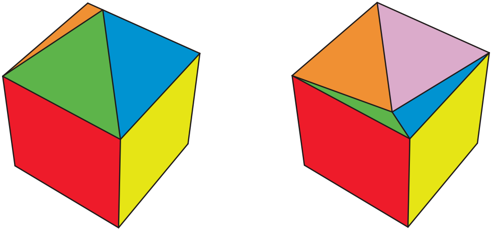
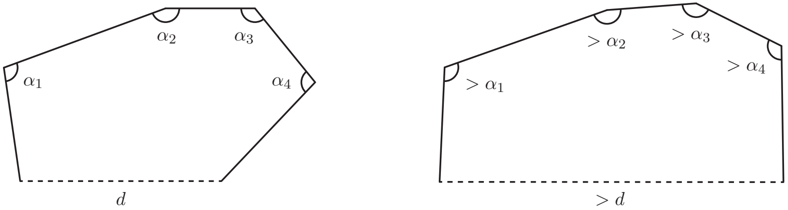


Heath, 1908: To be fair, Euclid only applies his definition to prove equality of <u>convex polyhedra with trihedral vertices</u>.

For these polyhedra, Euclid's statement is obviously true, because a trihedral vertex is rigid.

Convex polyhedra are rigid

Cauchy's arm lemma:



Theorem (Legendre-Cauchy, 1813)

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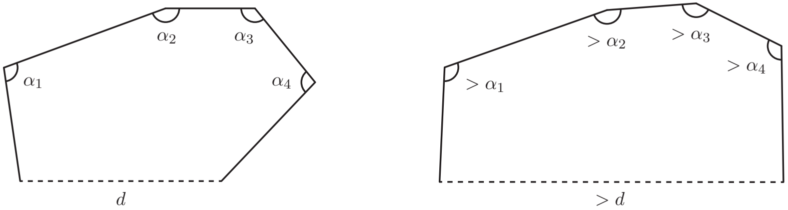

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Corollary

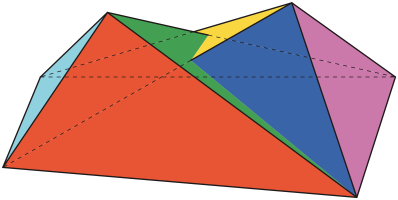
Convex polyhedra are rigid.

$\Longrightarrow$ Polyhedral bellows must be non-convex.

Existence of flexible polyhedra

Theorem (Bricard, 1897)

There exist self-intersecting flexible octahedra.



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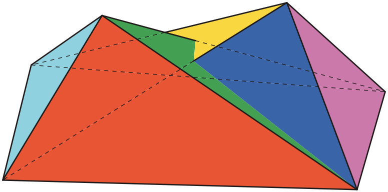

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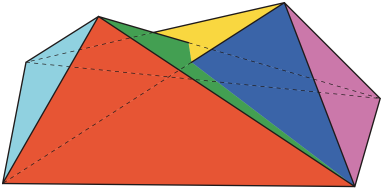

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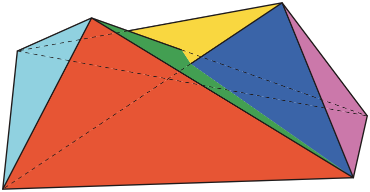

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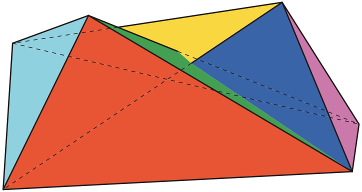

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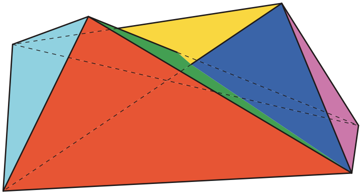

Theorem (Gluck, 1975)

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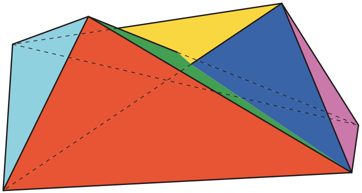


Theorem (Gluck, 1975)

Almost all polyhedra of genus 0 are rigid.

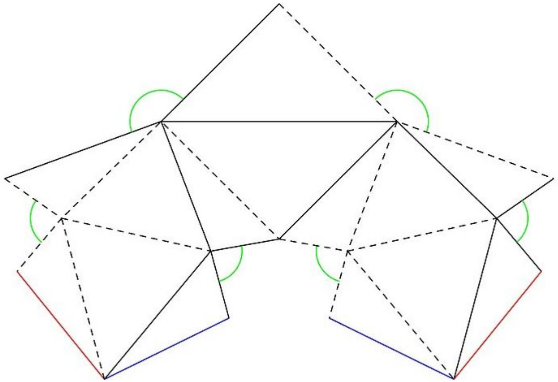
Theorem (Connelly, 1977)

There exist (non-self-intersecting) flexible polyhedra.

Steffen's flexible polyhedron

Theorem (Steffen, 1979)

There is a flexible polyhedron with 9 vertices (smallest possible).



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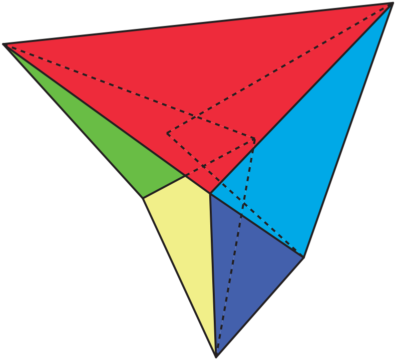

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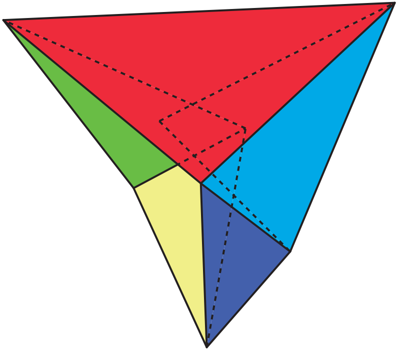

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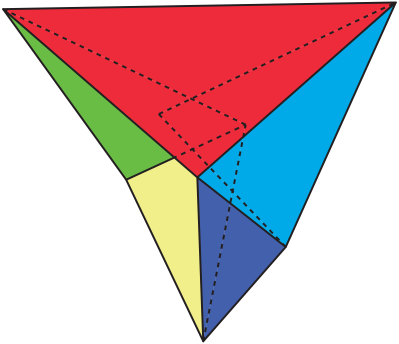

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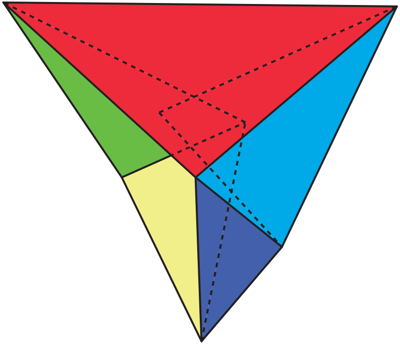

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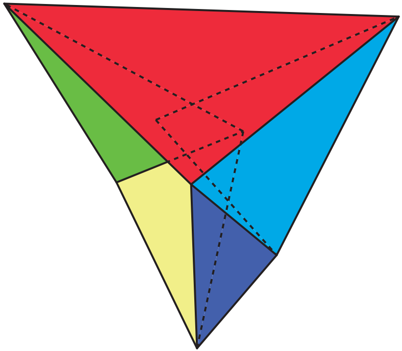

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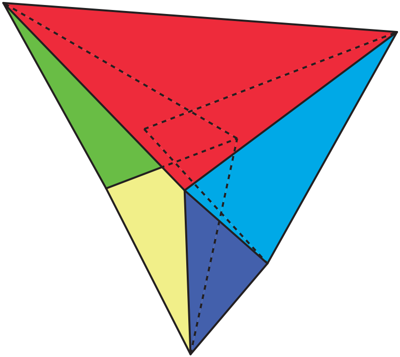

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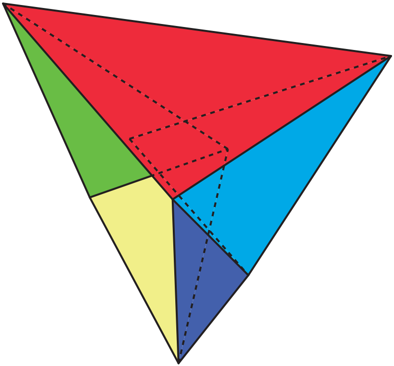

Still no polyhedral bellows!

Observation

The (generalized) <u>volume</u> of all these flexible polyhedra remains <u>constant</u> throughout the flexing!

In other words, althought these polyhedra are not rigid, none of them can blow air.

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Observation

The (generalized) volume of all these flexible polyhedra remains constant throughout the flexing!

In other words, althought these polyhedra are not rigid, none of them can blow air.

Several people made the conjecture that this is not a coincidence, and Connelly coined a name for it:

Bellows conjecture (Connelly, 1978)

The volume of a polyhedron is constant throughout any flexing. In other words, there are no polyhedral bellows.

Bellows theorem

Theorem (Sabitov, 1996)

Given a polyhedron (of any genus), its volume V satisfies

$$V^N + a_{N-1}(\bar{\ell})V^{N-1} + \cdots + a_1(\bar{\ell})V + a_0(\bar{\ell}) = 0,$$

where the coefficients $a_i(\bar{\ell})$ only depend on the combinatorial structure of the polyhedron and on the lengths of its edges, $\bar{\ell}$.

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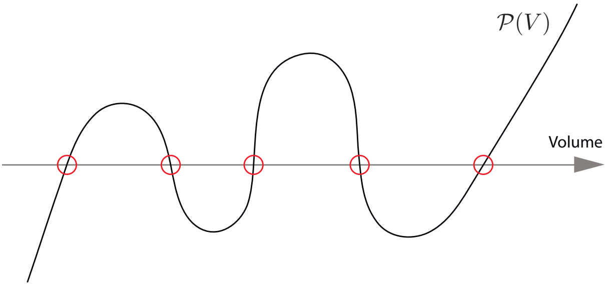
The volume is a root of a polynomial that remains fixed as the polyhedron flexes, hence it can only take finitely many values.

Corollary (Bellows theorem)

The volume of a polyhedron is constant throughout any flexing.

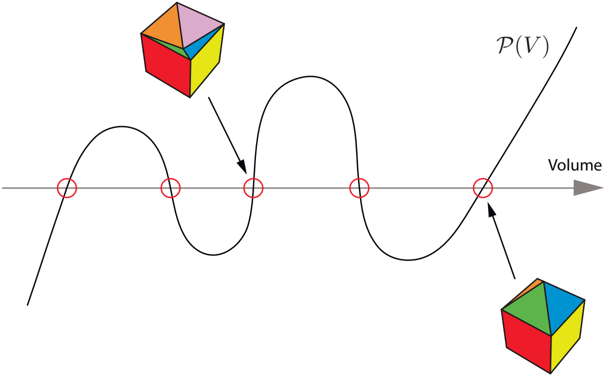
$\Longrightarrow$ There are no polyhedral bellows (of any genus).

Bellows theorem

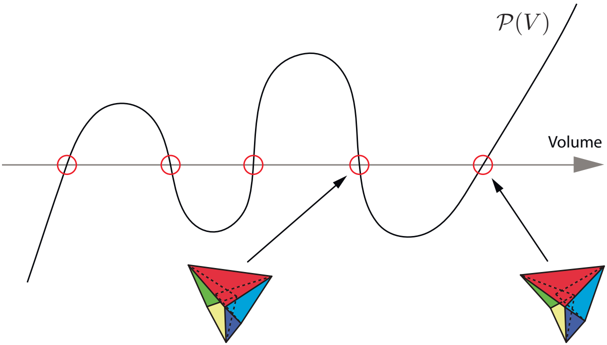


Any polynomial has a finite number of roots:
at most as many as its degree.

Bellows theorem



So, the volume of a polyhedron with assigned combinatorial structure and edge lengths can only take finitely many values.



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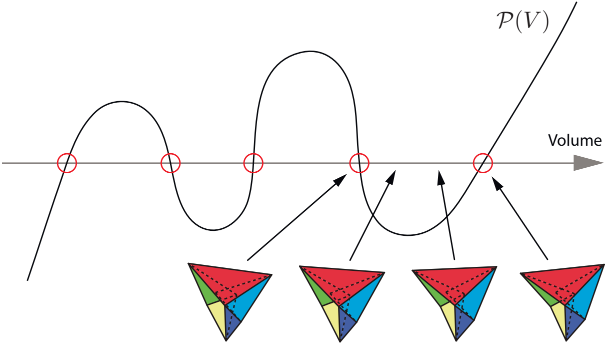

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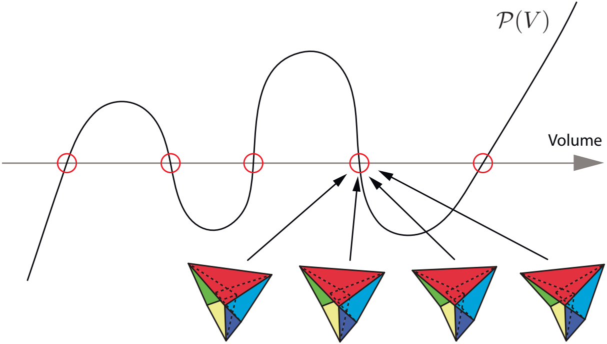

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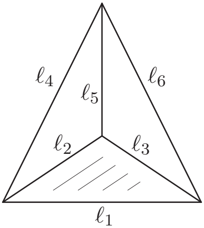
Example: tetrahedra

There is an analogue of Heron's formula for tetrahedra:

Theorem (Piero della Francesca, 15th century)

The volume V of a tetrahedron with edges $\ell_1, \cdots, \ell_6$ satisfies

$$V^2 = \frac{1}{144}\big[\ell_1^2\ell_5^2(\ell_2^2+\ell_3^2+\ell_4^2+\ell_6^2-\ell_1^2-\ell_5^2) + \ell_2^2\ell_6^2(\ell_1^2+\ell_3^2+\ell_4^2+\ell_5^2-\ell_2^2-\ell_6^2) + \ell_3^2\ell_4^2(\ell_1^2+\ell_2^2+\ell_5^2+\ell_6^2-\ell_3^2-\ell_4^2) - \ell_1^2\ell_2^2\ell_3^2 - \ell_2^2\ell_4^2\ell_5^2 - \ell_1^2\ell_4^2\ell_6^2 - \ell_3^2\ell_5^2\ell_6^2\big]$$



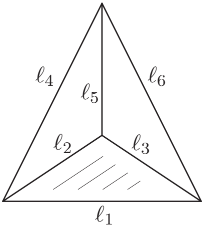
Example: tetrahedra

There is an analogue of Heron's formula for tetrahedra:

Theorem (Piero della Francesca, 15th century)

The volume V of a tetrahedron with edges $\ell_1, \cdots, \ell_6$ satisfies

$$V^2 = \frac{1}{144}\big[\ell_1^2\ell_5^2(\ell_2^2+\ell_3^2+\ell_4^2+\ell_6^2-\ell_1^2-\ell_5^2)+\ell_2^2\ell_6^2(\ell_1^2+\ell_3^2+\ell_4^2+\ell_5^2-\ell_2^2-\ell_6^2)$$
$$+\ell_3^2\ell_4^2(\ell_1^2+\ell_2^2+\ell_5^2+\ell_6^2-\ell_3^2-\ell_4^2)-\ell_1^2\ell_2^2\ell_3^2-\ell_2^2\ell_4^2\ell_5^2-\ell_1^2\ell_4^2\ell_6^2-\ell_3^2\ell_5^2\ell_6^2\big]$$



The polynomial equation has the form $\mathcal{P}(V) = V^2 + a_0(\bar{\ell}) = 0$.

Generalizations and extensions

Theorem (Gaifullin, 2012)

The bellows theorem generalizes to any dimension > 3.

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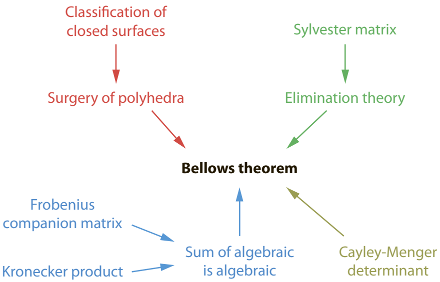
~~*The strong bellows conjectures is false.*~~ *Wrong?*

Theorem (Gaifullin-Ignashchenko, under review)

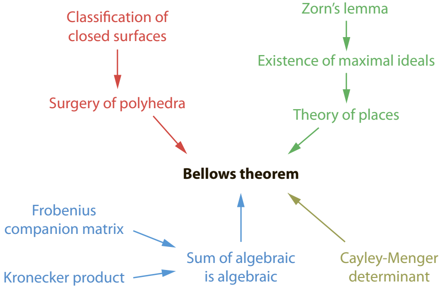
A polyhedron preserves its Dehn invariant throughout any flexing. Hence the strong bellows conjecture is true. (cf. Sydler, 1965)

Next seminar: proving the bellows theorem

Geometric approach



Algebraic approach



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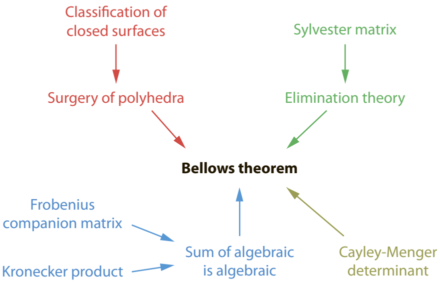


Pros:

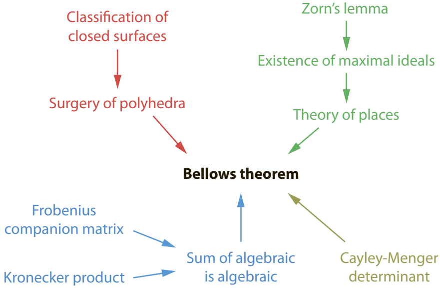
Volume polynomial explicitly constructed

Self-contained proof
(given basic linear algebra and topology)

Cons:

Some cumbersome computations

Algebraic approach



Cons:

Non-contructive proof

Heavier on theory (valuation rings, extension of homomorphisms, ...)

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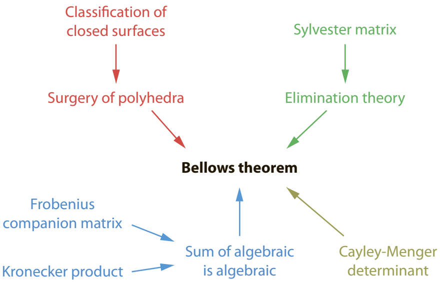
Next seminar: proving the bellows theorem

Geometric approach



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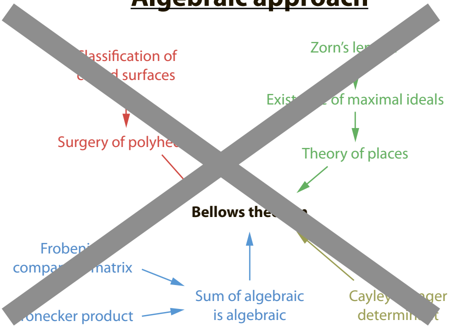

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Next seminar: prerequisites

Determinant of a square matrix:

- Basic properties, e.g., $\det(AB) = \det(A)\det(B)$
- How to compute it, especially by Laplace expansion
- Usage in linear algebra: A is invertible iff $\det(A) \neq 0$
- Geometric interpretation: scaling factor of the linear transformation described by the matrix

Eigenvalues of a square matrix:

- Definition and basic properties
- Relationship with the characteristic polynomial of the matrix

Closed orientable surfaces:

- Geometric intuition of genus: number of holes
- Topological intuition of surgery: what happens when a circular cut is made on a closed orientable surface