Guarding and Searching Polyhedral Environments

Giovanni Viglietta

Department of Computer Science, University of Pisa, Italy

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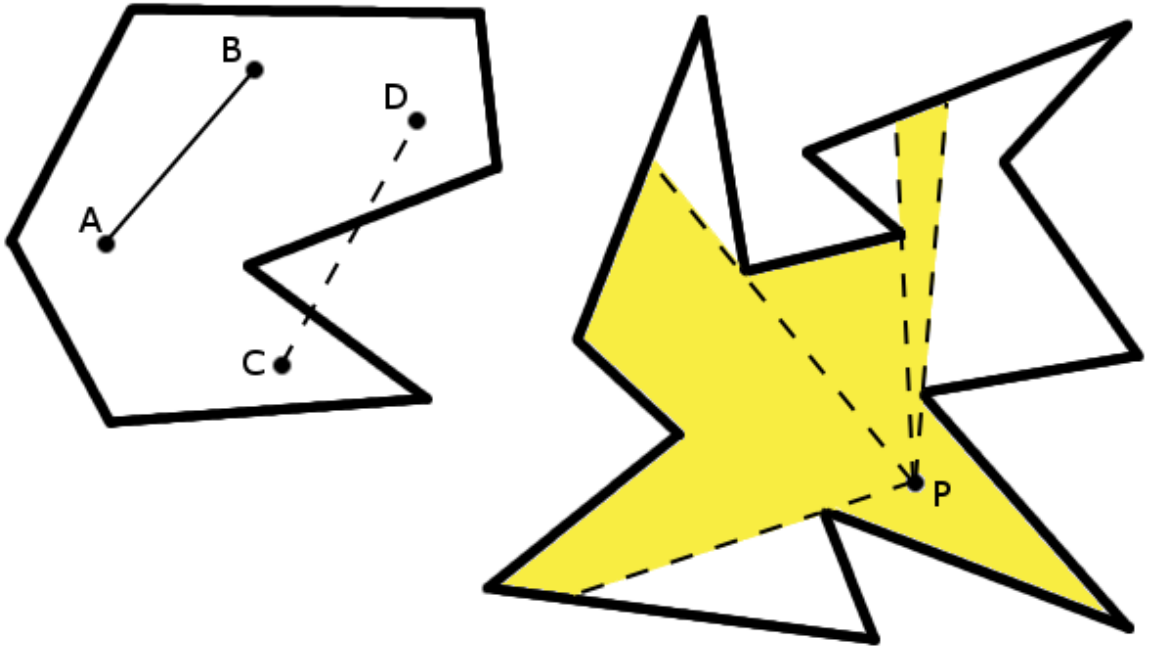
Euclidean visibility

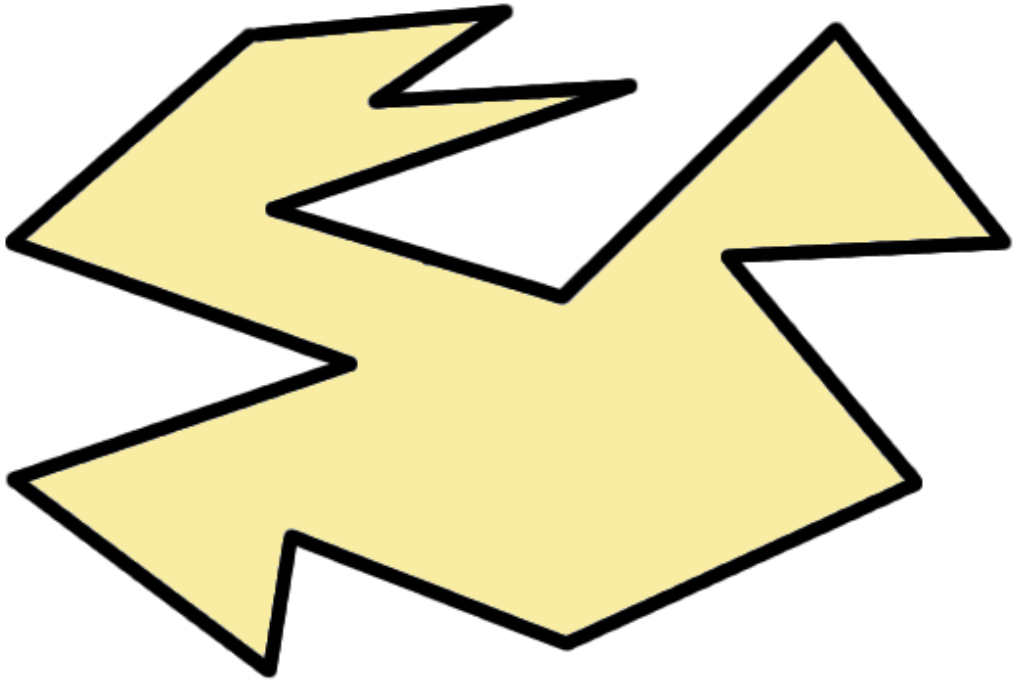
Visibility is a central concept in robotics, motion planning, CAD, CAM, pattern recognition, etc..



Visibility problems are well-studied in the plane, but there's a general lack of results concerning their 3-dimensional counterparts.

Art Gallery Problem - triangulations

Fisk, 1973: How many guards should we place in a simple n-polygon, in order to see its whole interior?



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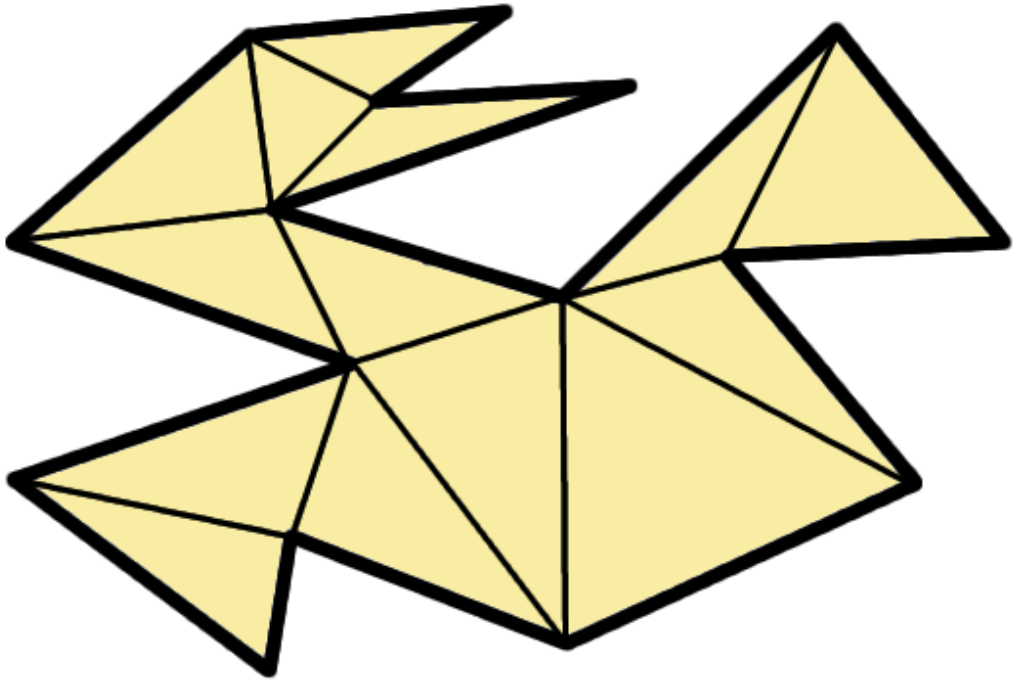

- Triangulation $\implies$ n vertex-guards are sufficient.

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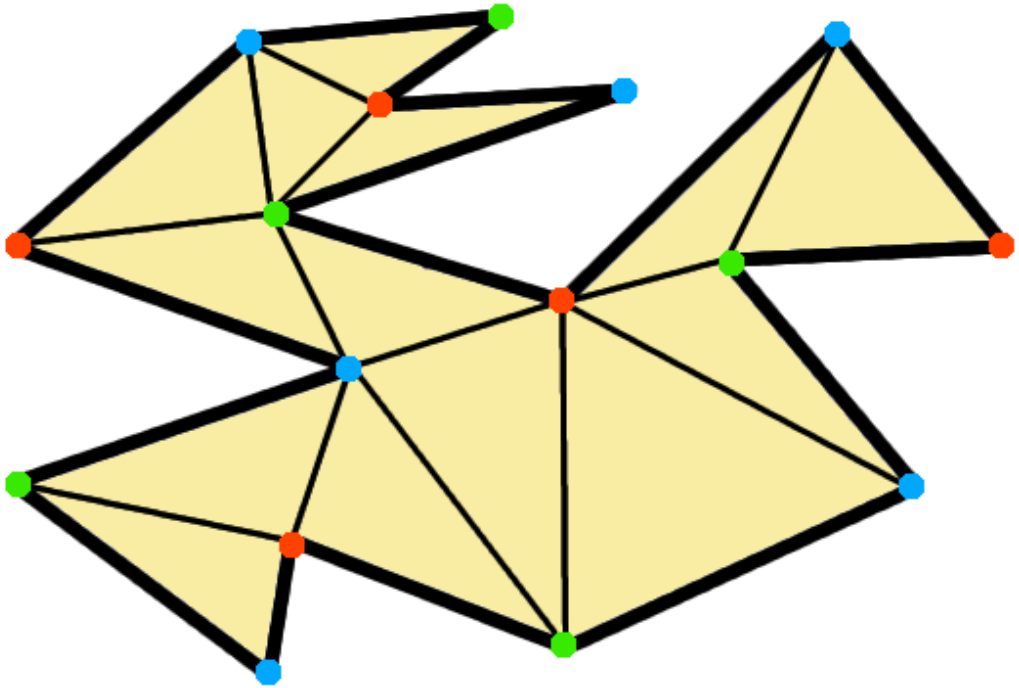
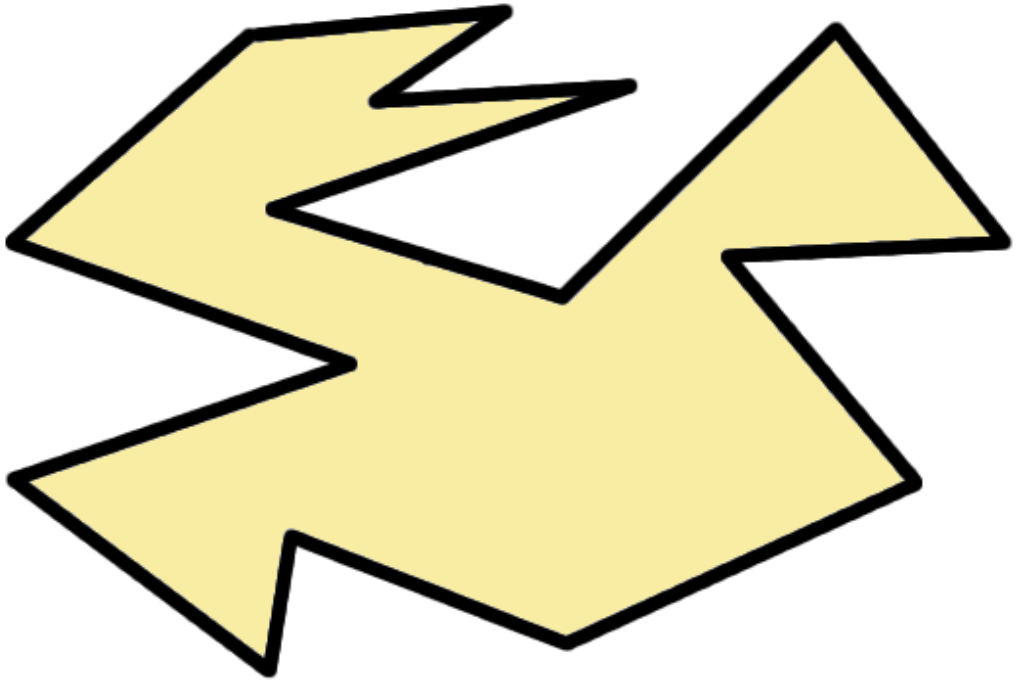


- Triangulation $\implies$ n vertex-guards are sufficient.
- **Chvátal, 1975:** $\lfloor \frac{n}{3} \rfloor$ vertex-guards are sufficient.

Art Gallery Problem - reflex vertices

Convexity can be exploited.



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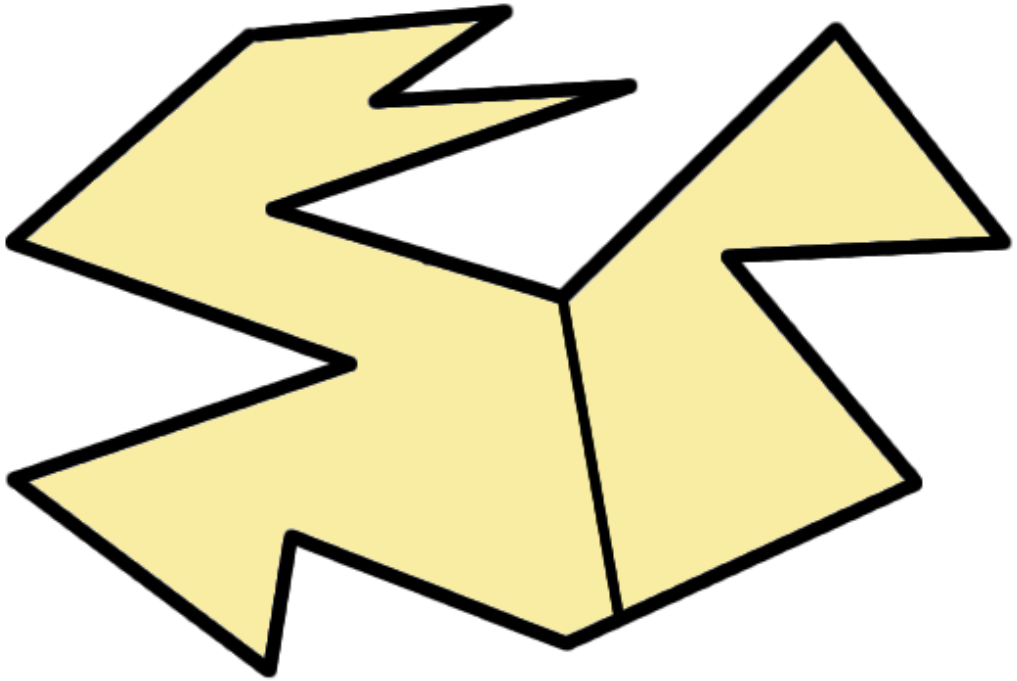

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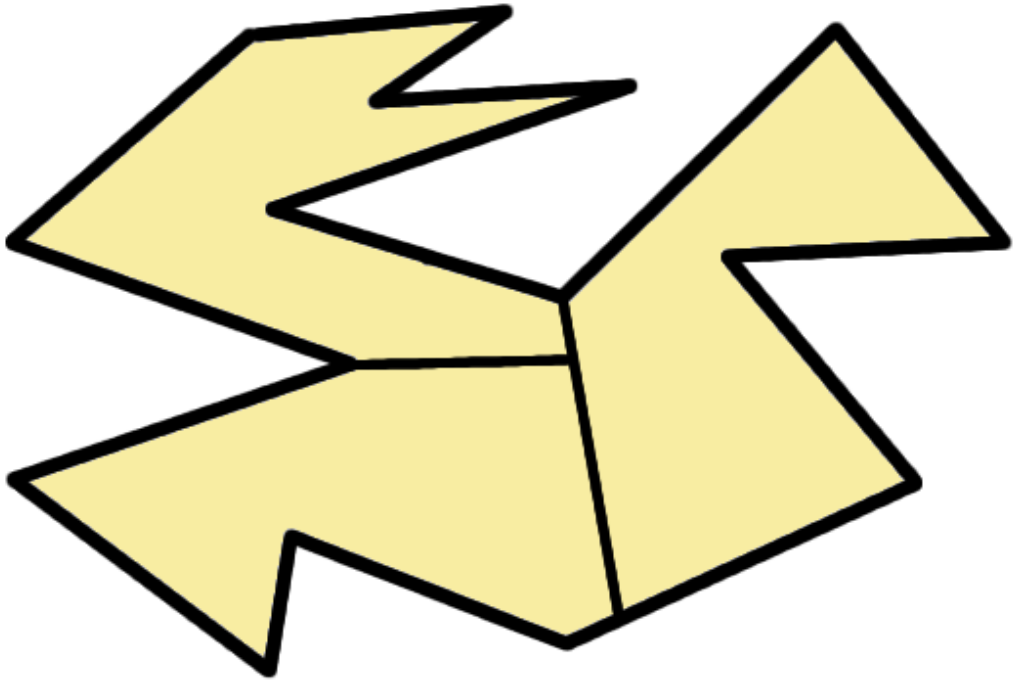

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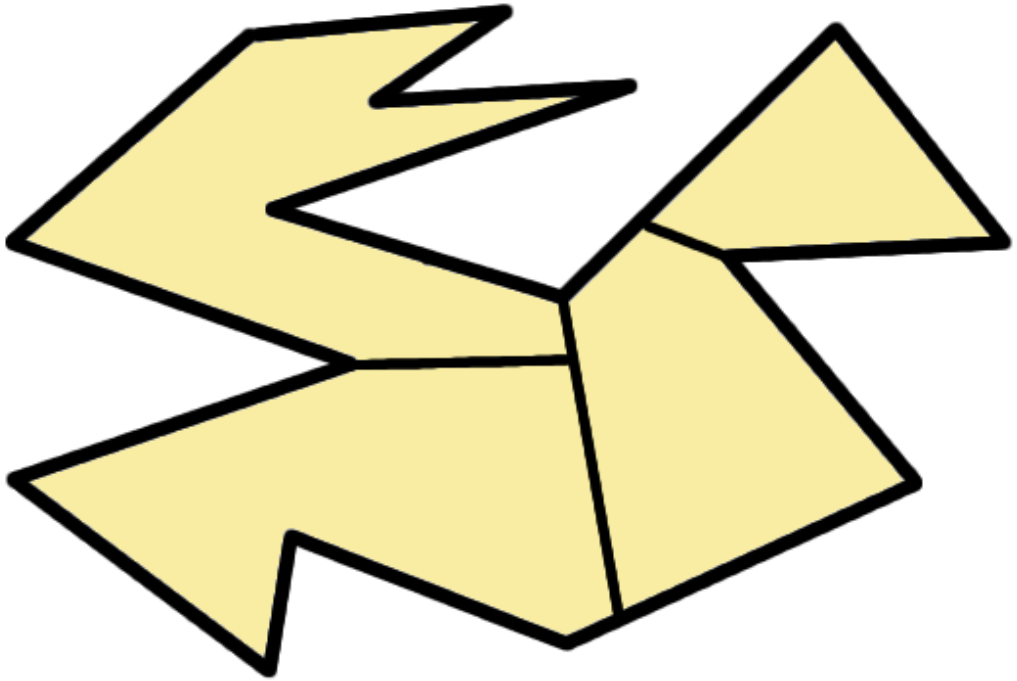

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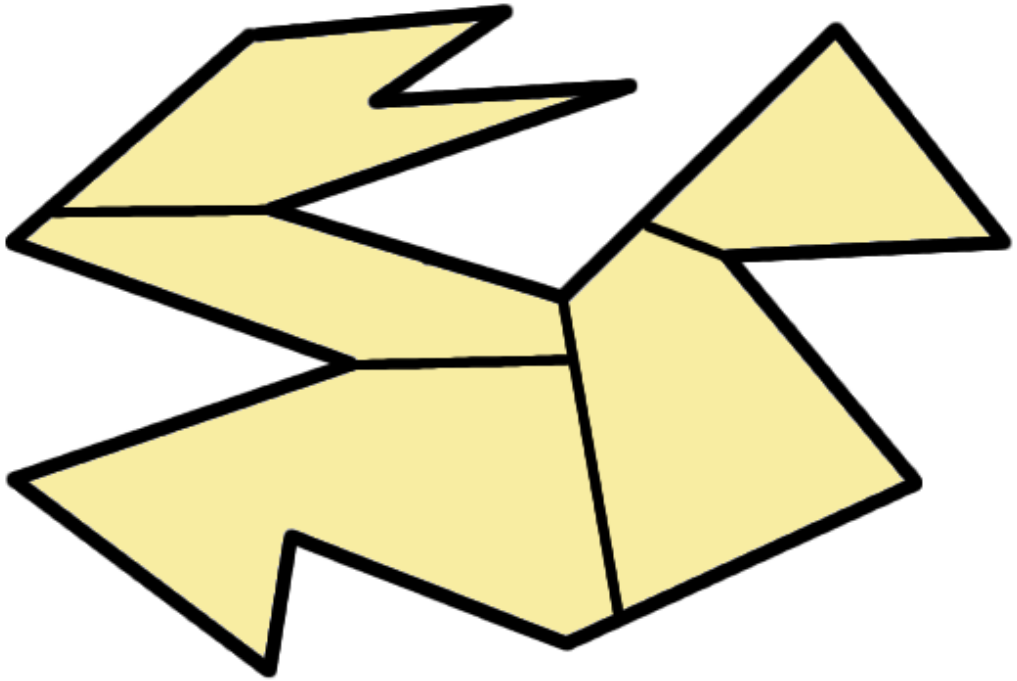

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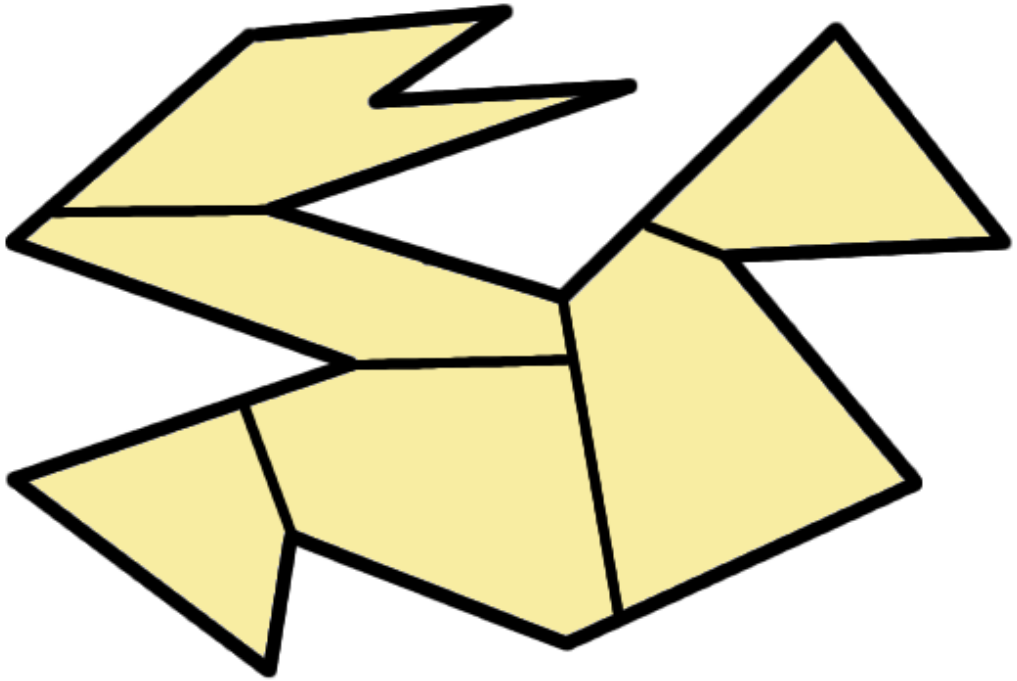

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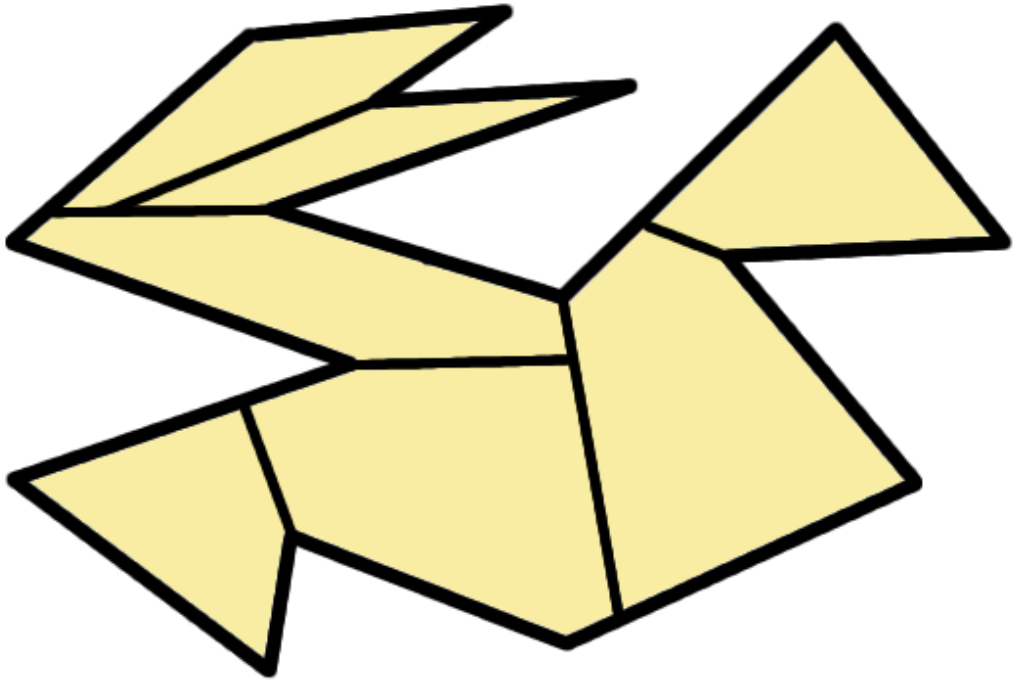

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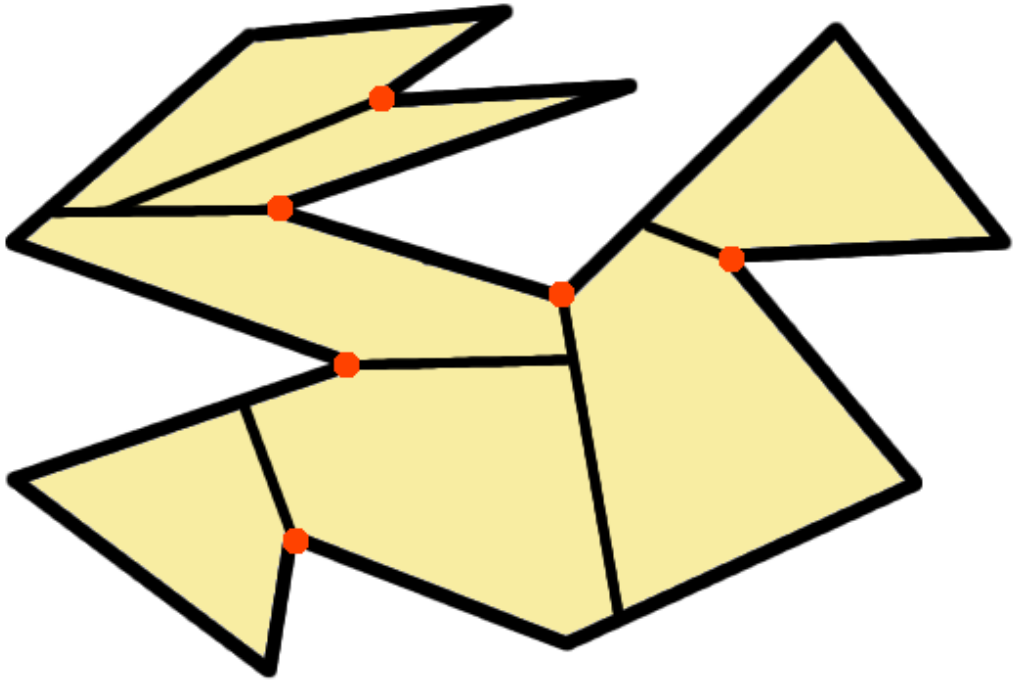

- **O'Rourke:** Guards can be placed at reflex vertices. Occasionally it's a better heuristic, occasionally it's not.

Art Gallery Problem - reflex vertices

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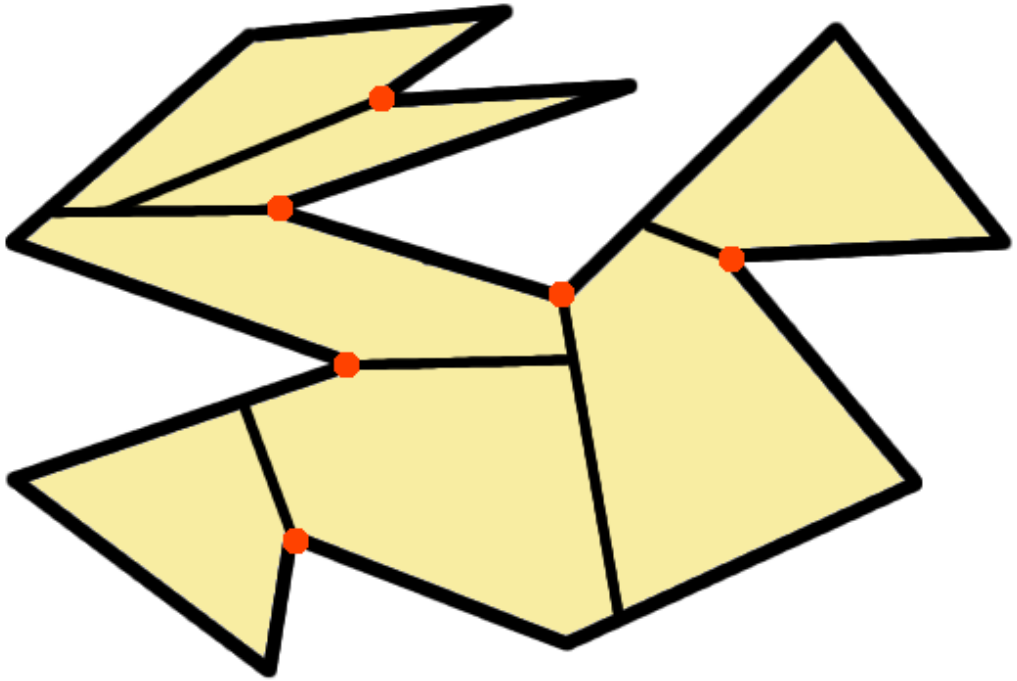
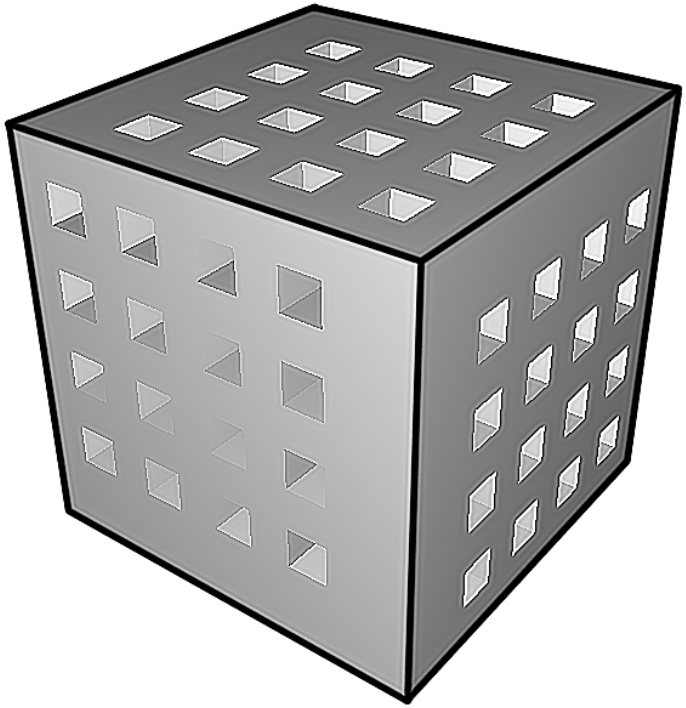
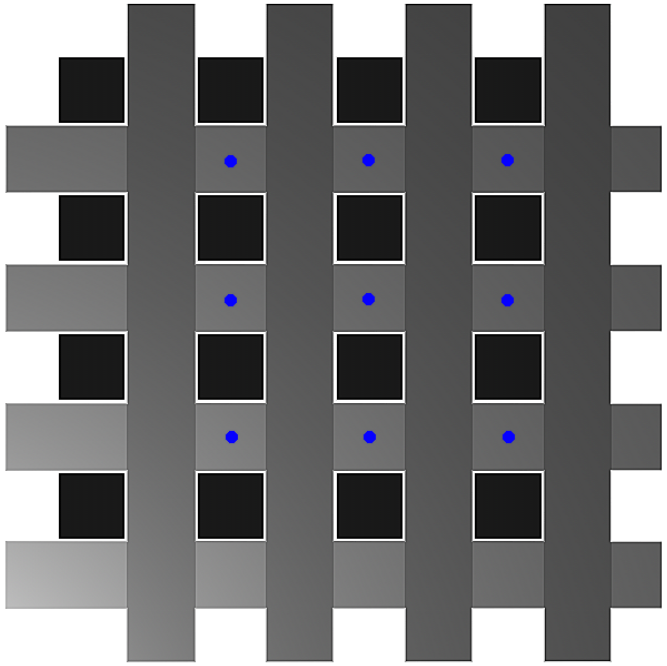


- **O'Rourke:** Guards can be placed at reflex vertices. Occasionally it's a better heuristic, occasionally it's not.
- **Lee & Lin, 1986:** Determining the minimum number of guards is NP-hard.

3D point-guards

Point-guards seem inadequate to watch polyhedra.

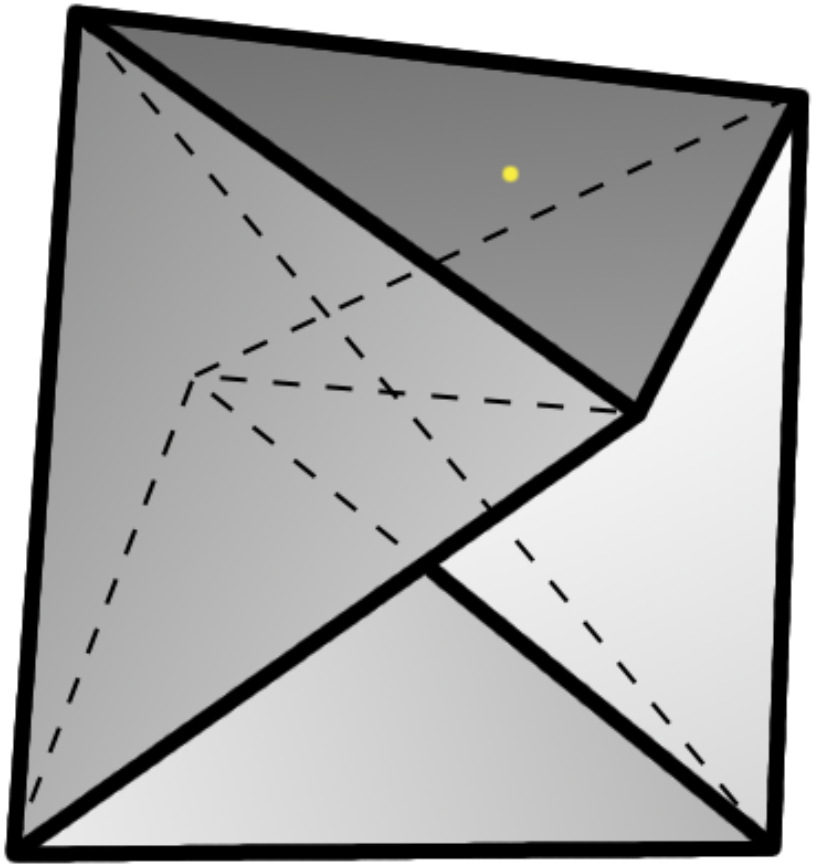




- Vertex-guards are not enough to see the interior.
- Partition into tetrahedra requires additional vertices.
- $\Omega(n^{3/2})$ point-guards are necessary.

3D line-guards

Linearity of the required guards is restored by considering edge-guards: every point of a polyhedron can see some edge.



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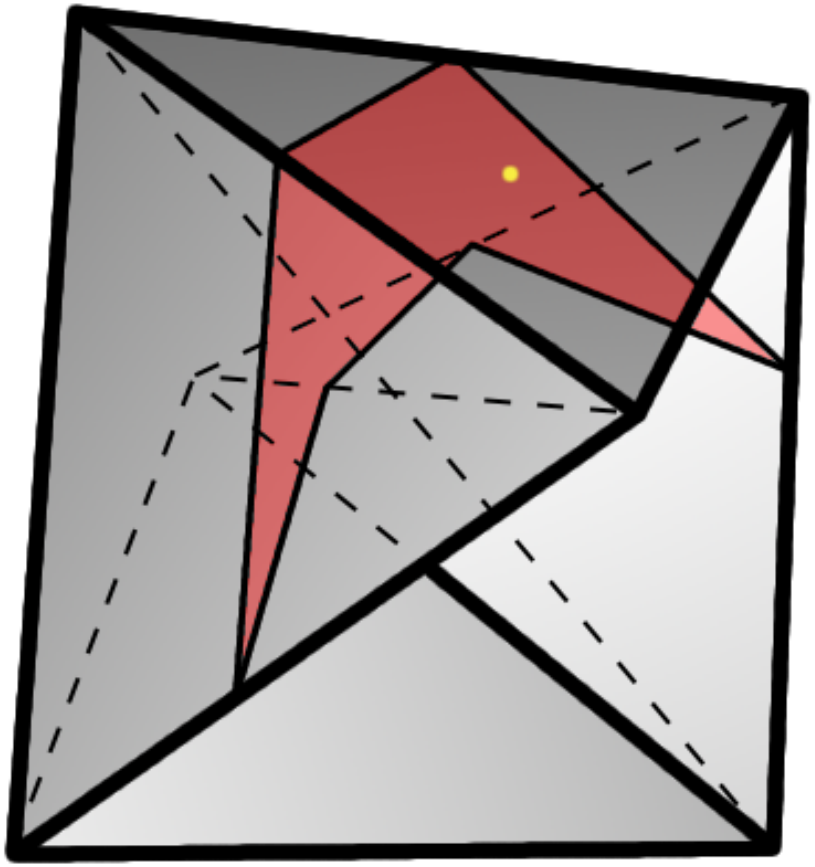

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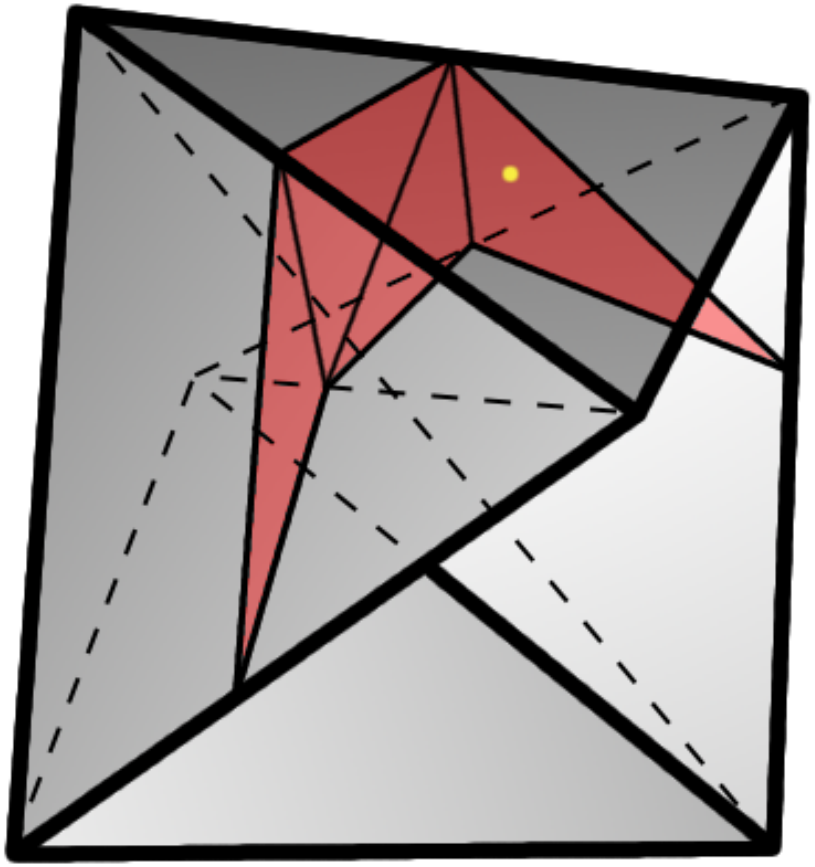
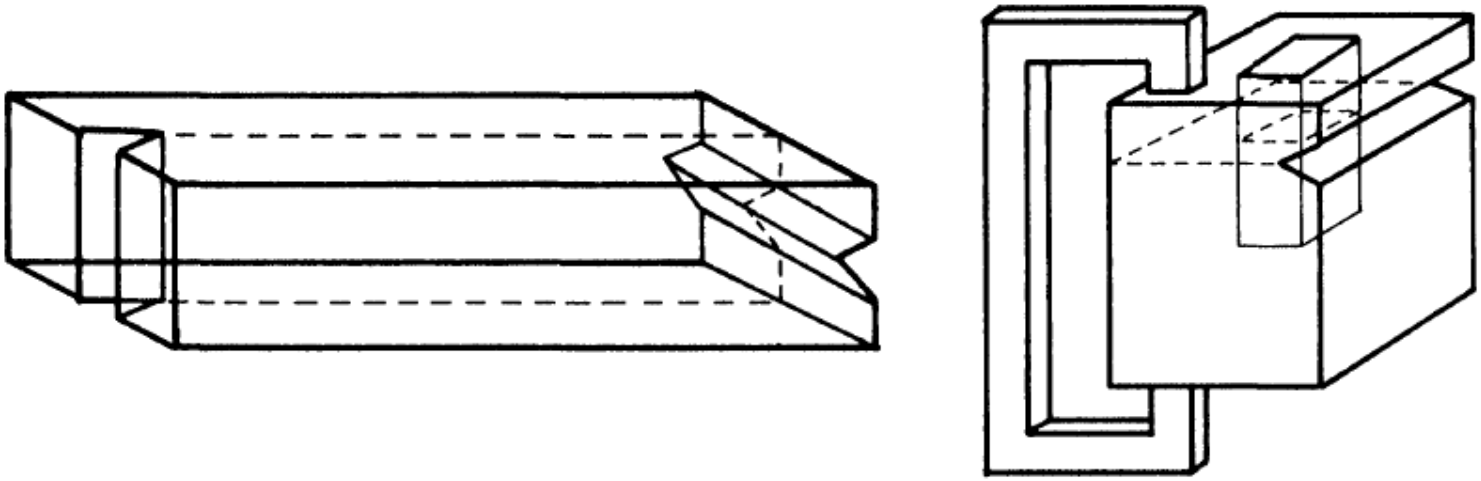


Exploiting reflex edges

Are reflex edges sufficient?
Attempting to generalize O'Rourke's construction leads to technical difficulties.



- Cuts along reflex edges may split other reflex edges.
- Cuts may increase the polyhedron's genus.
- Cuts may fail to disconnect the polyhedron.

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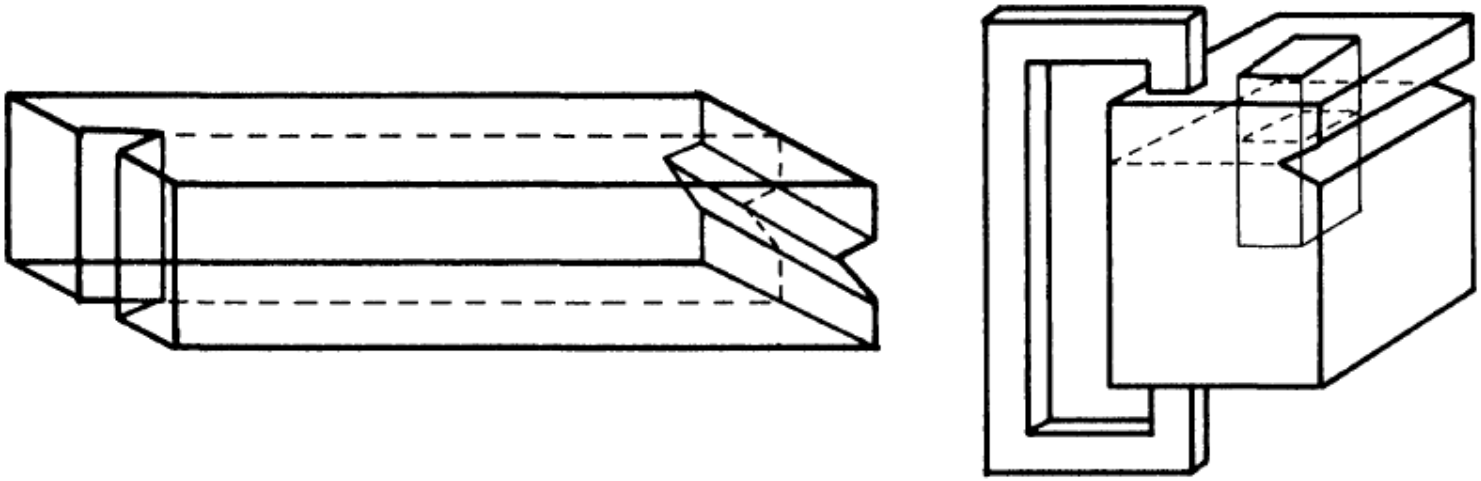

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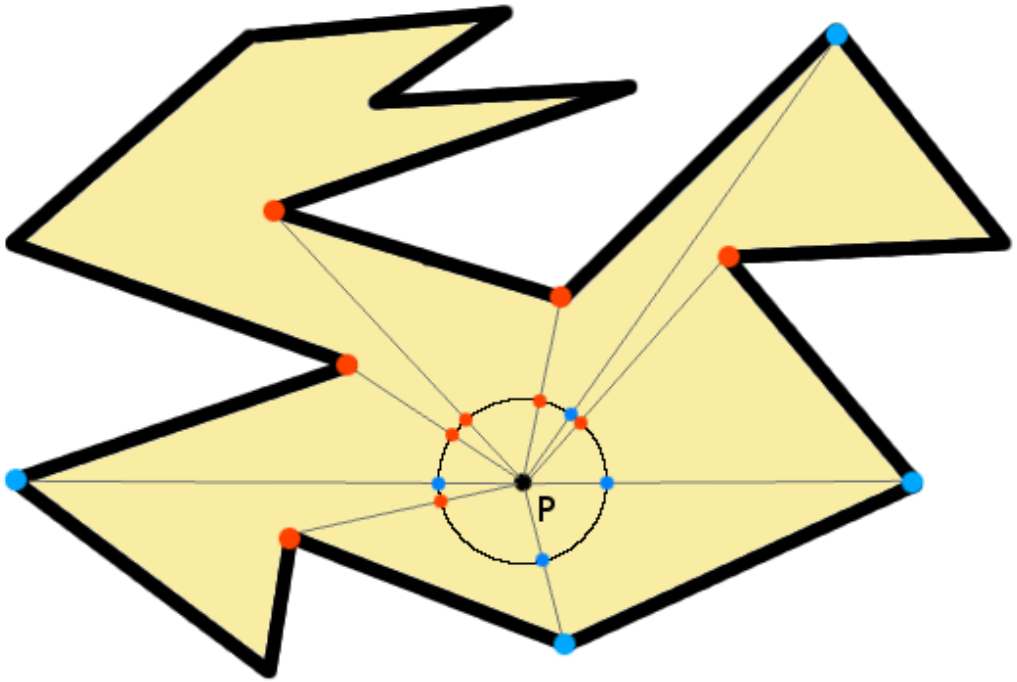
The proof can still be carried out with this method, but a different approach seems more natural...

2D view graphs

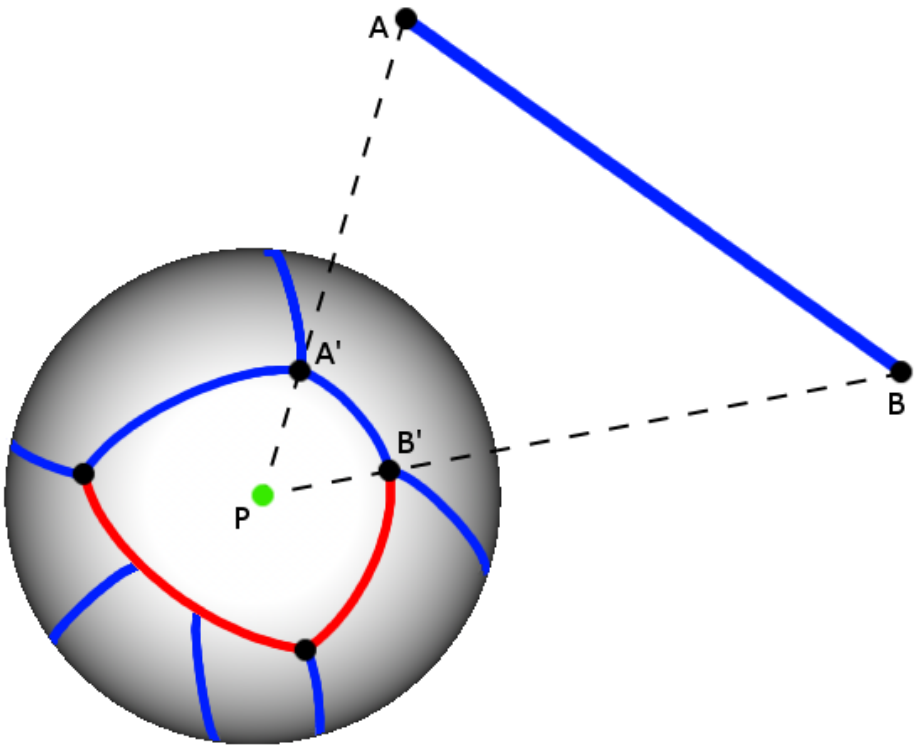
Introducing view graphs.



All nodes of the view graph of a point P come from convex vertices if and only if the whole polygon boundary is visible from P.

3D view graphs

3D view graphs carry significant information, and may compensate for the lack of tetrahedralizations.



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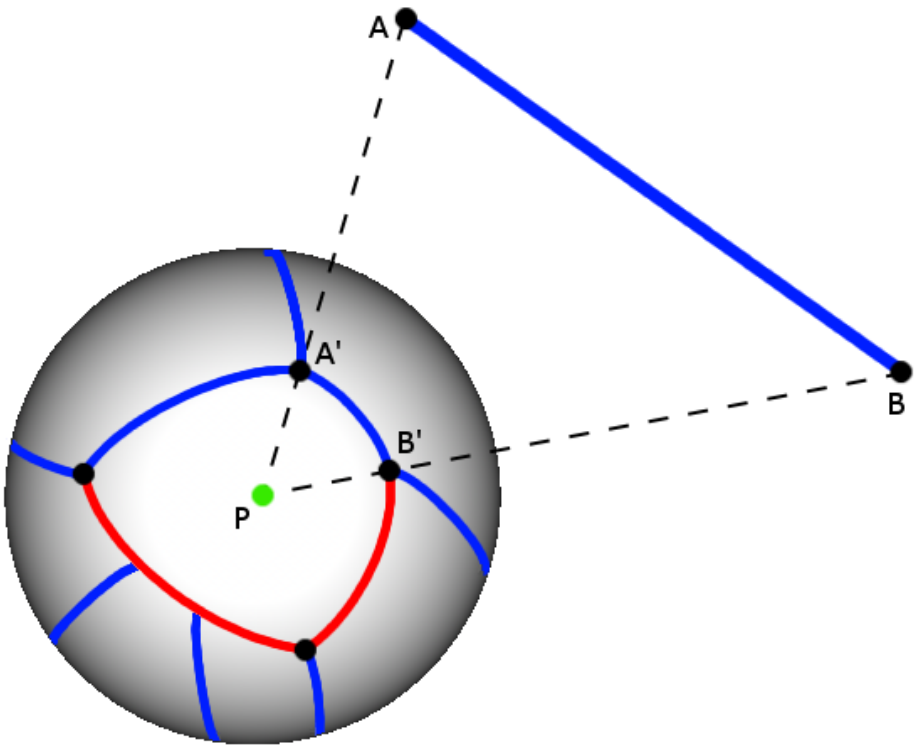
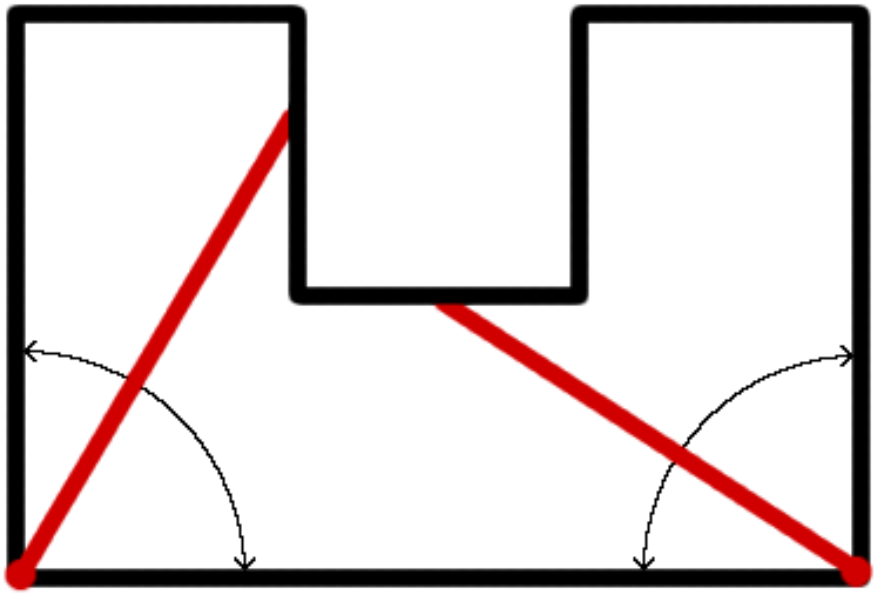


All edges of the view graph of a point P come from convex edges if and only if the whole polyhedron boundary is visible from P.

$\Longrightarrow$ Reflex edges see the whole polyhedron.

Searchlight Scheduling Problem

Suzuki & Yamashita, 1990: Suppose the guards can only see through an orientable searchlight, and the intruder can run arbitrarily fast. Does a searchlight movement schedule exist that always catches the intruder?



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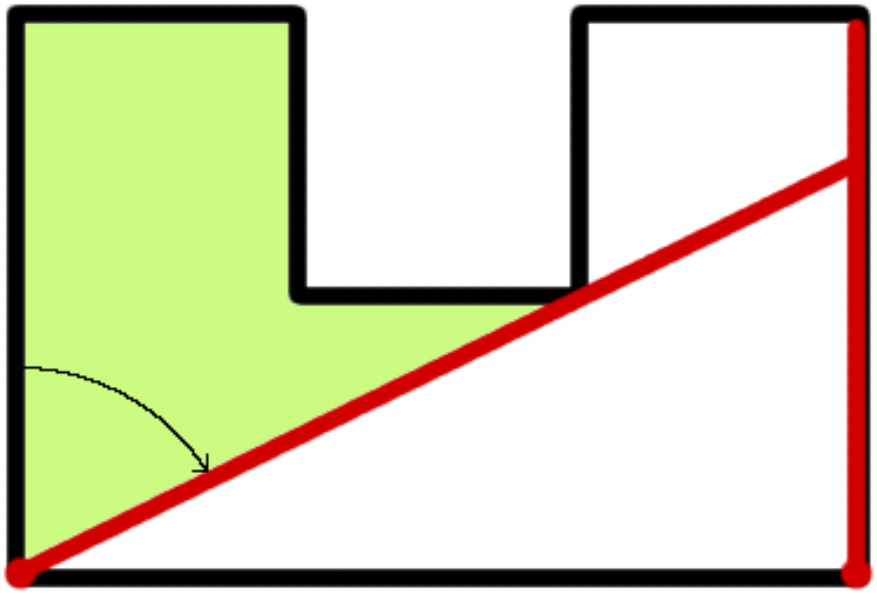

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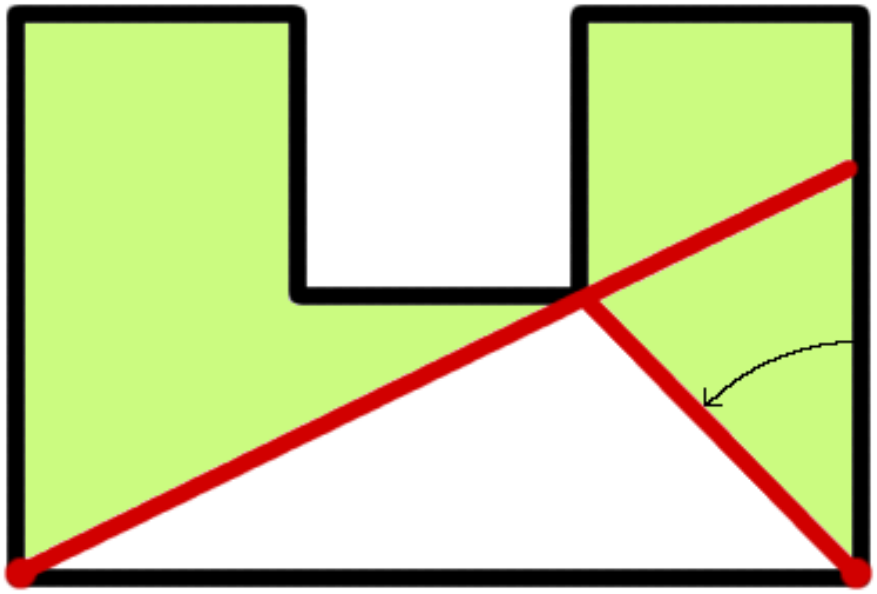

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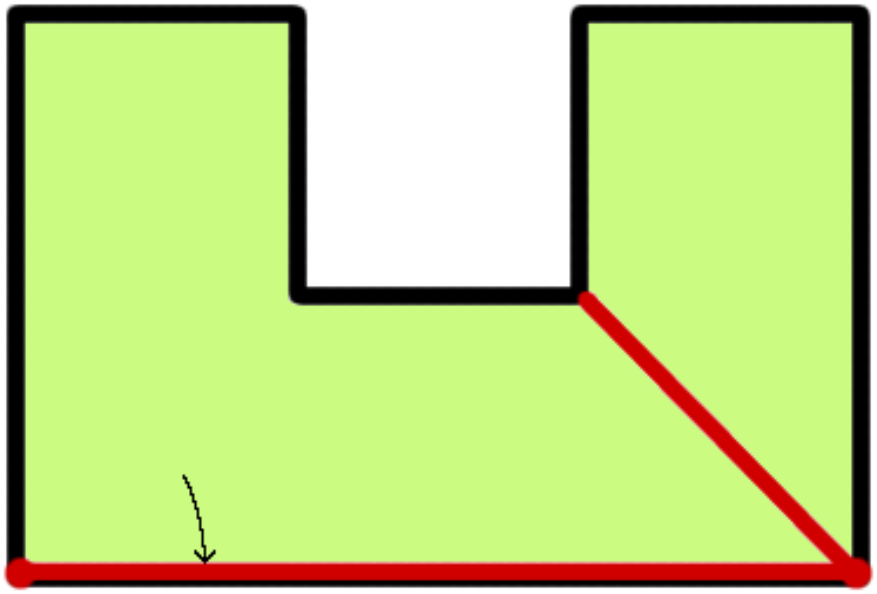
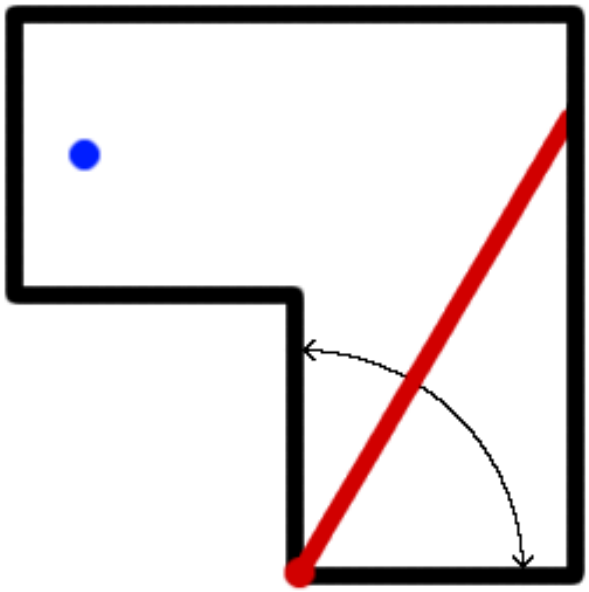
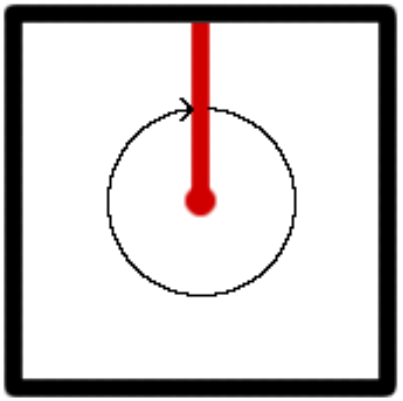


Solvability of 2SSP

Some instances are trivially unsolvable.





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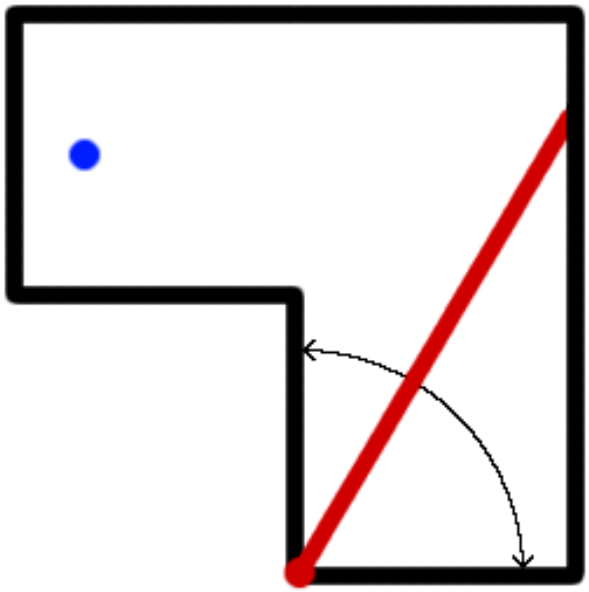
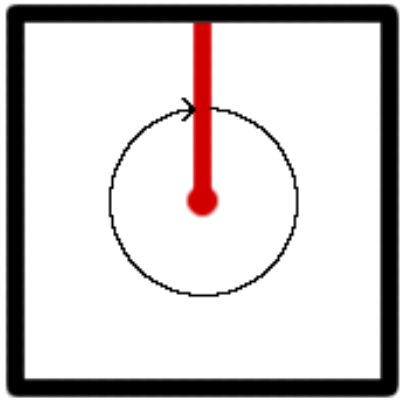


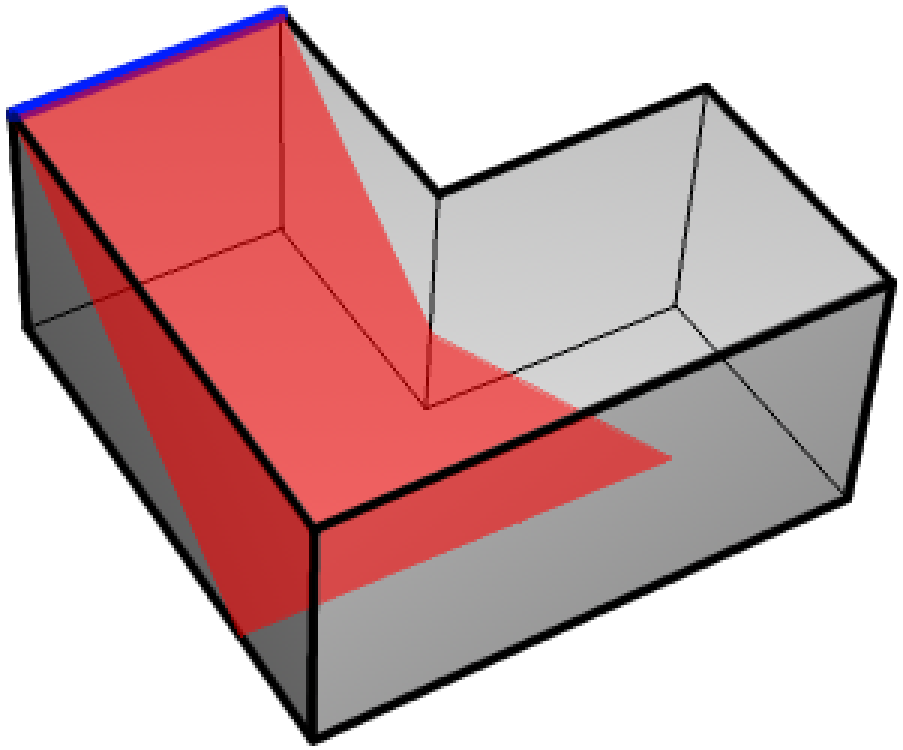


If all guards lie on the border and every point in the polygon can be seen by at least one guard, then the instance is solvable.

2SSP: Is a given instance solvable?
Determining if 2SSP is NP-hard is an important open problem.

3D searchlights

3SSP: Guards are segments contained in a polyhedron, and searchlights are half-planes emanating from such segments.



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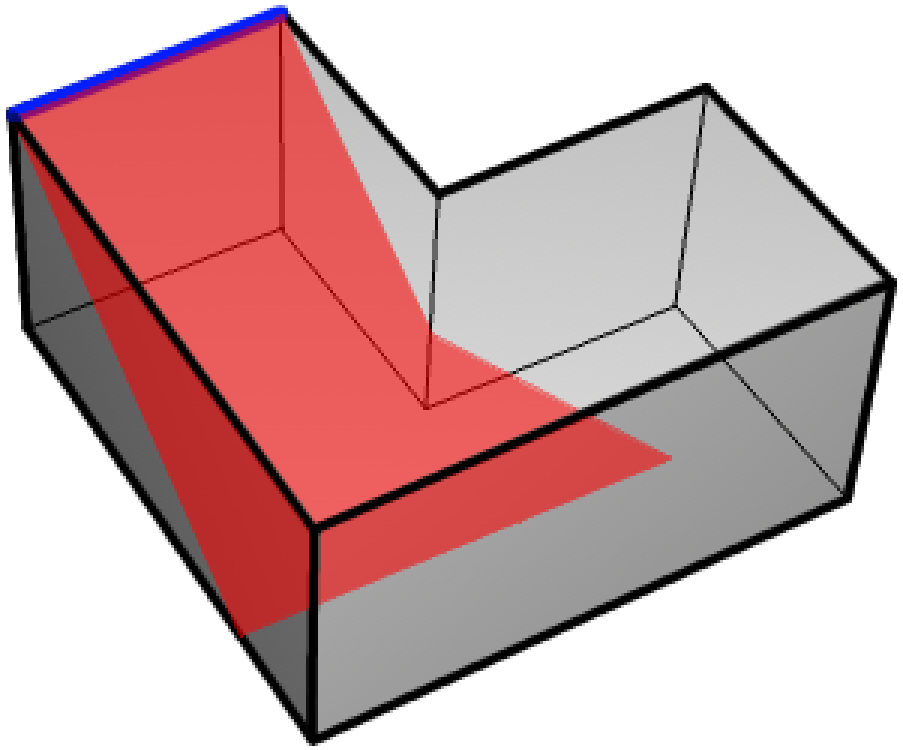
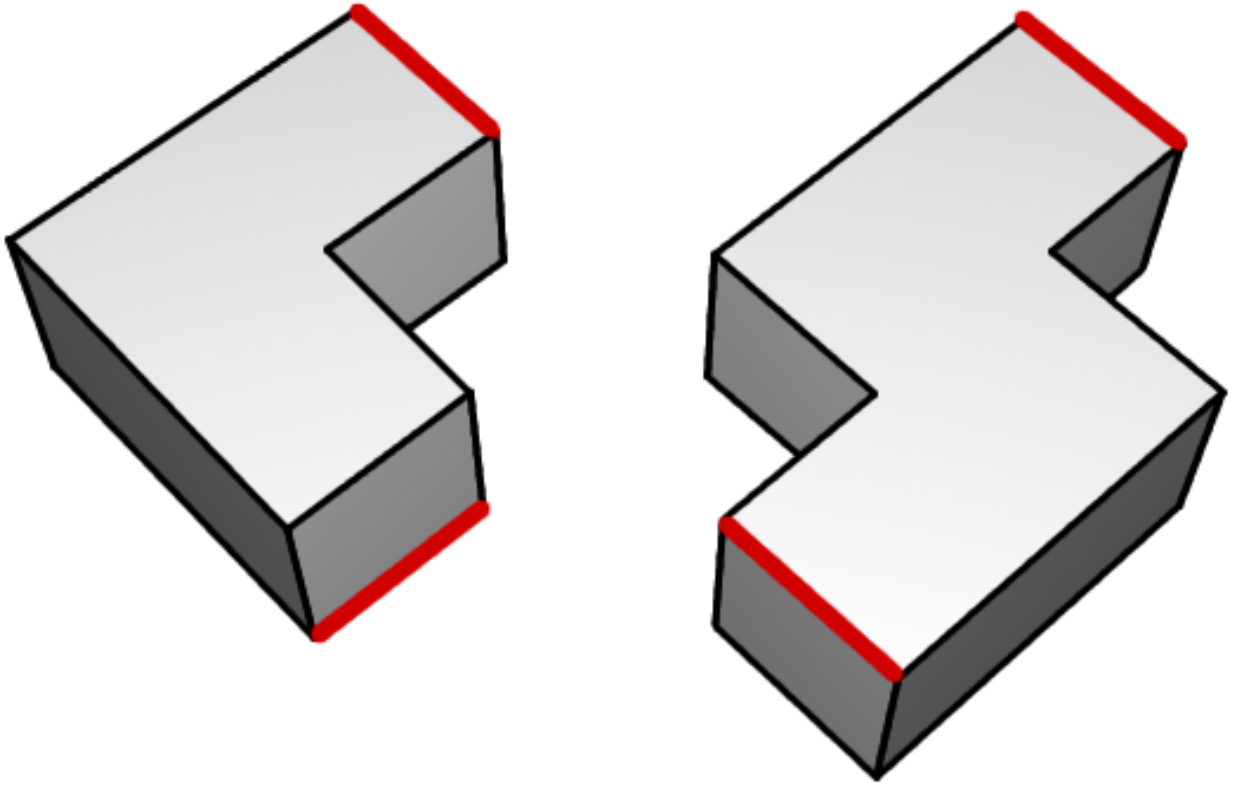


- All the advantages of edge-guards.
- Searchlights still have 1 degree of freedom.
- 2SSP $\preceq_{\mathbf{P}}$ 3SSP.

More unsolvable instances

Boundary guards may fail to solve an instance, even if they can see the whole interior.



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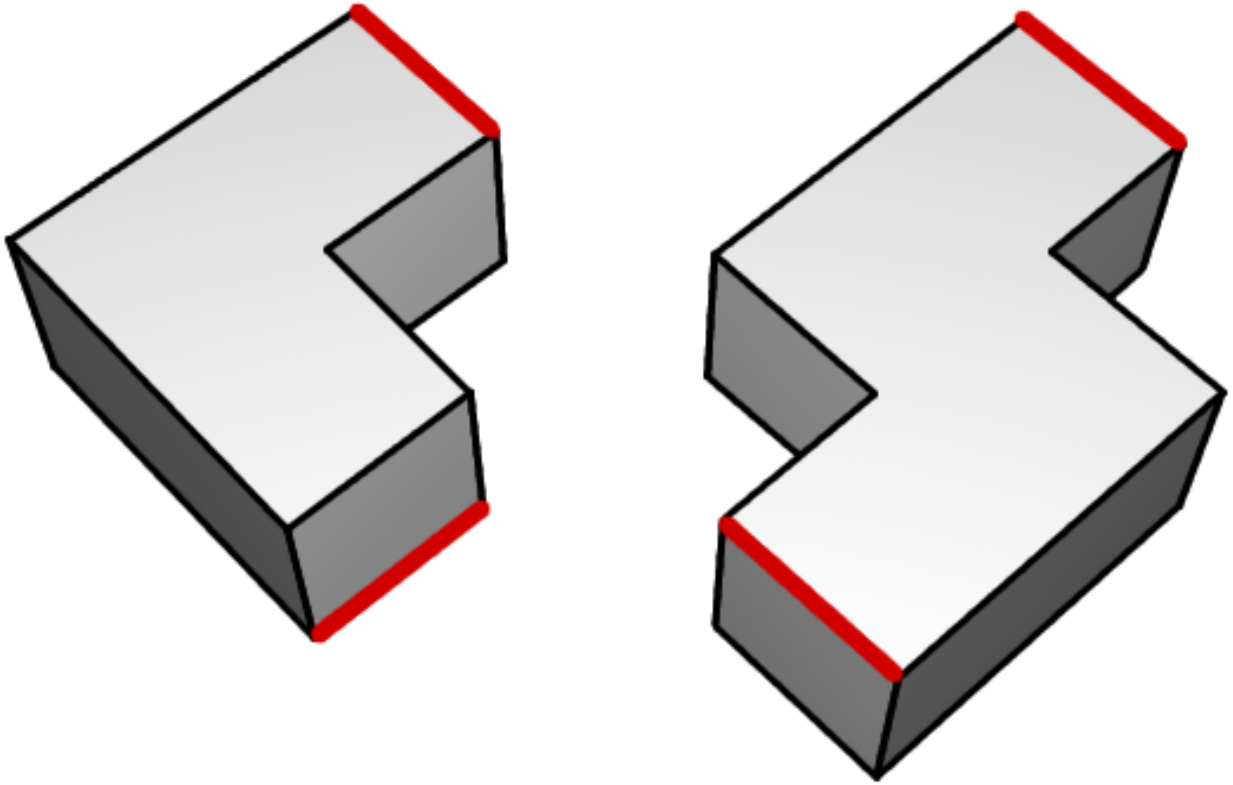

However, instances with only one guard are solvable if and only if the guard lies on the boundary and can see the whole interior.

Timed 3SSP is NP-hard

T3SSP: Decide if a 3SSP instance is solvable within a given time, where searchlights have bounded angular speed.

Claim: 3SAT $\preceq_{\mathbf{P}}$ T3SSP.

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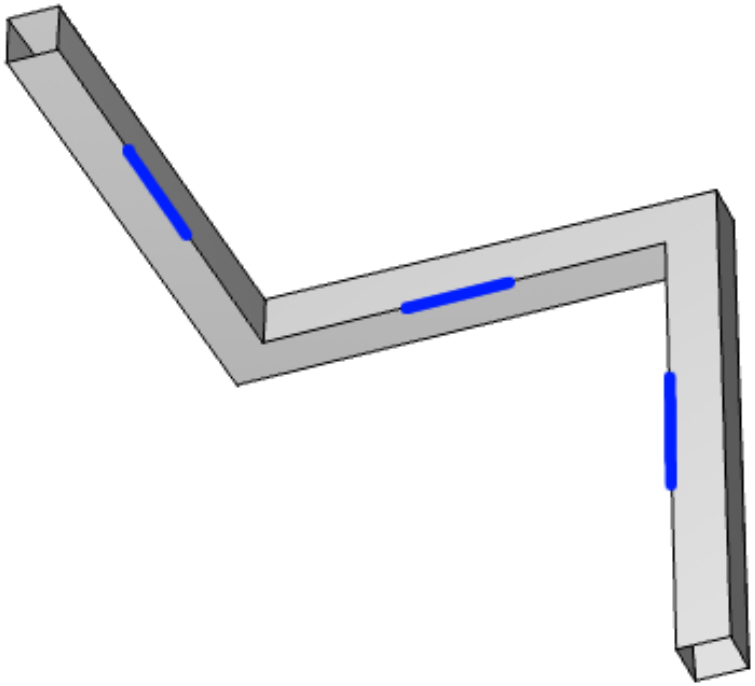
Let's convert a 3CNF formula F into an instance of 3SSP which is solvable in 3 seconds if and only if F is satisfiable, where the angular speed of guards is $90^\circ/\,\text{sec}$.

Moreover:

- the polyhedron may be chosen to be orthogonal,
- its size is linear in the size of F,
- all the guards may be chosen to lie on edges,
- all coordinates are integers.

Orthogonal link

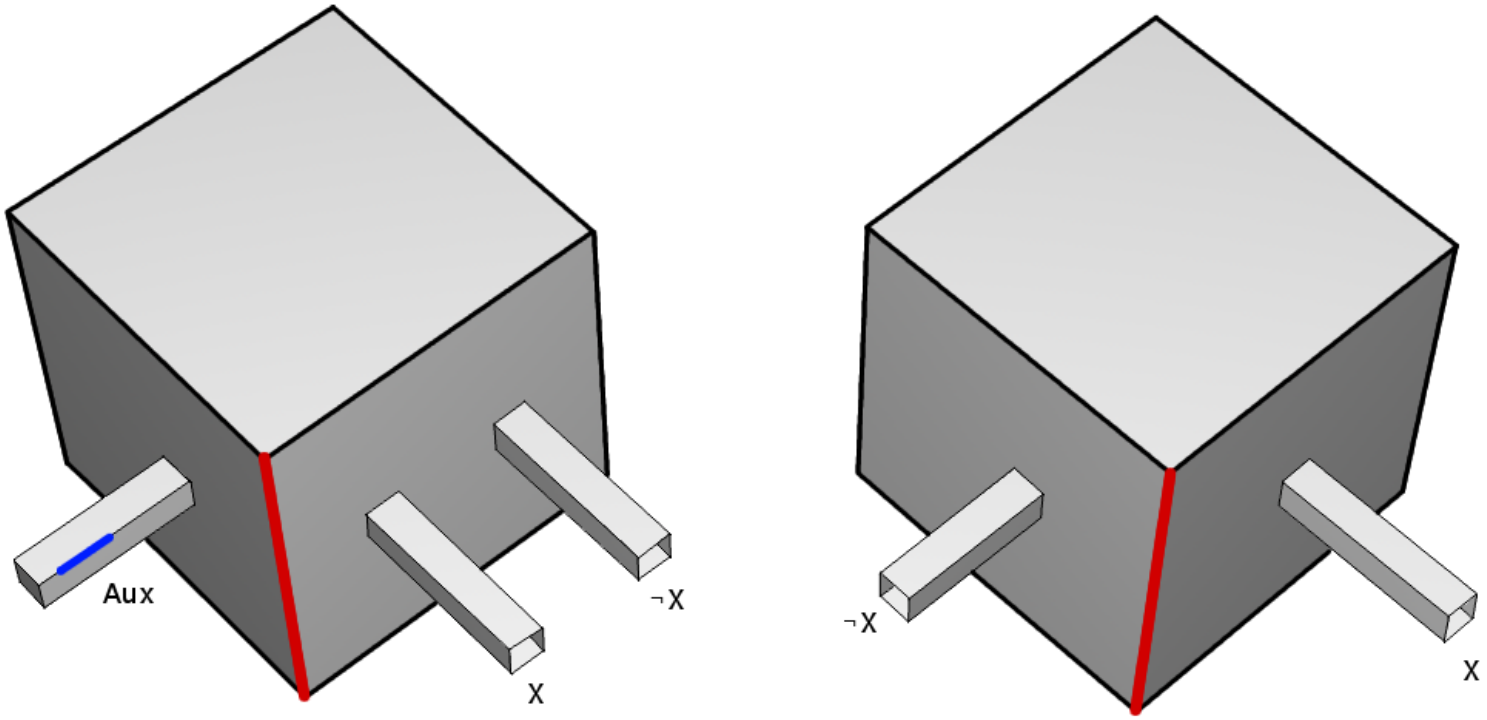
A link can be cleared in 1 second, if and only if both its ends are capped.



It may also be bent at will in all (orthogonal) directions, so that guards are forced to turn simultaneously.

Boolean variable

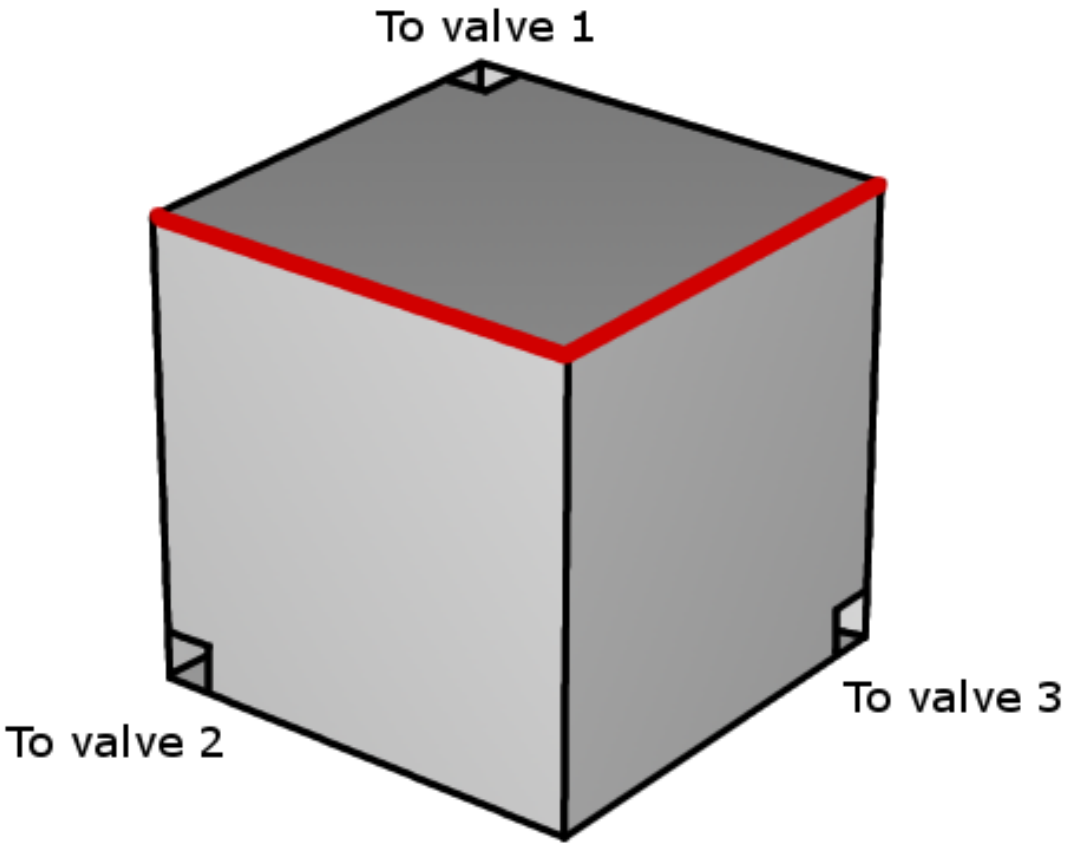
Since the solvability of T3SSP instances is invariant with respect to time inversions in the whole schedule, a boolean value must stay the same after such inversions.



The variable is true if and only if both red guards turn in the same direction.

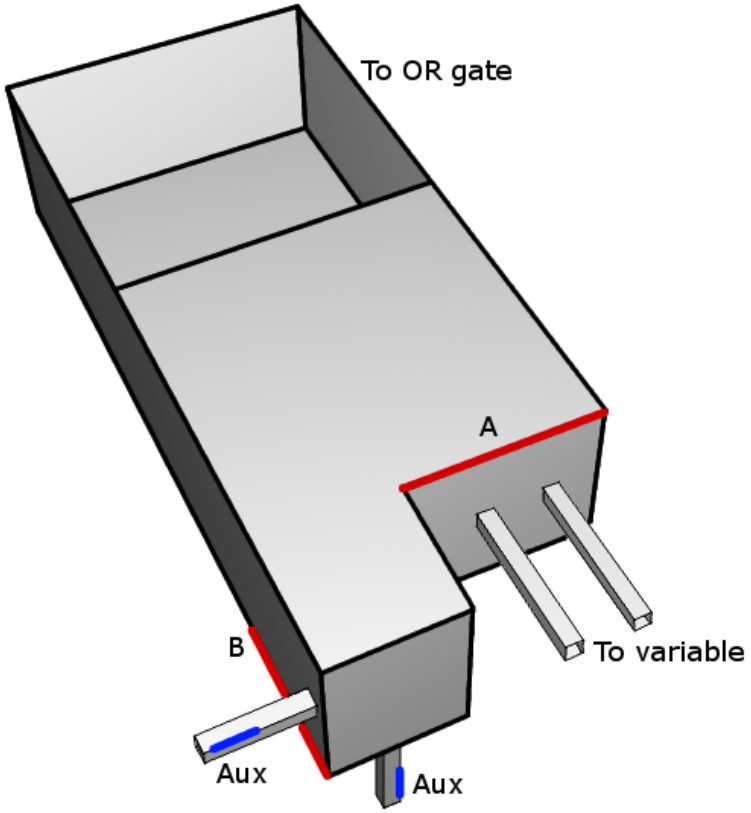
3-input OR gate

The cube can be cleared if and only if at least one of the valves is closed.



Valve

If the variable is true, both links can be cleared simultaneously, and guard A can turn to close the valve.
If the variable is false, guard A is forced to cap the links, and can't close the valve.



In both cases, guard B clears the valve during second 2.

Further research

- Design linear size decompositions for broad-enough classes of (orthogonal) polyhedra.
- Study 3D view graphs.
- Characterize instances of 3SSP solvable by 2 or 3 guards.
- Prove hardness of approximation of T3SSP.
- Prove that 3SSP is NP-hard (without time constraints).
- Prove that 2SSP is NP-hard.
- Design distributed heuristics for 3SSP.

Thank you!

References

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