A Theory of Spherical Diagrams

CCCG 2022

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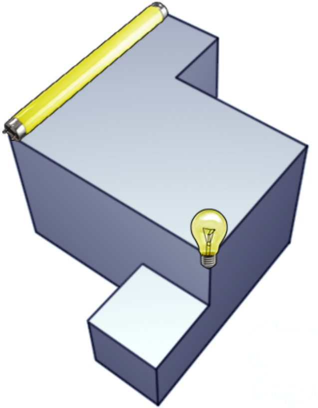
3D Art Gallery Problem

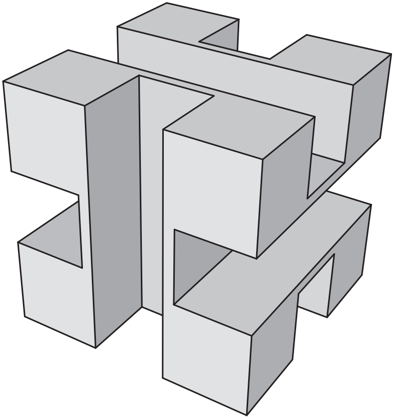
Given a polyhedron in $\mathbb{R}^3$, choose a (preferably small) set of vertices or edges that collectively see its whole interior.



These are called **vertex guards** and **edge guards**.

Vertex-guarding polyhedra

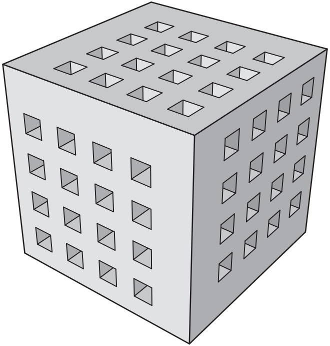
The Art Gallery Problem for *vertex guards* may be unsolvable, even in some orthogonal polyhedra:



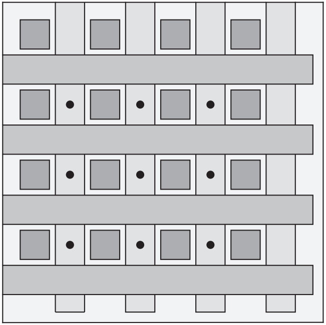
Some points in the central region are invisible to all vertices!

Point-guarding polyhedra

Even if guards are not constrained to lie on vertices, there are (orthogonal) polyhedra that require $\Omega(n\sqrt{n})$ *point guards*!



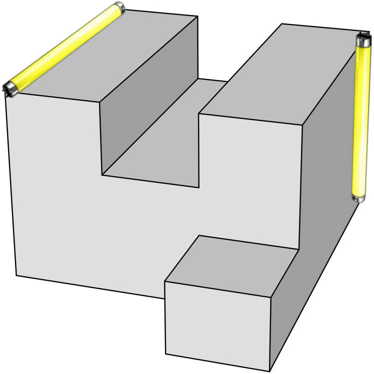
outer view



cross section

Edge-guarding polyhedra

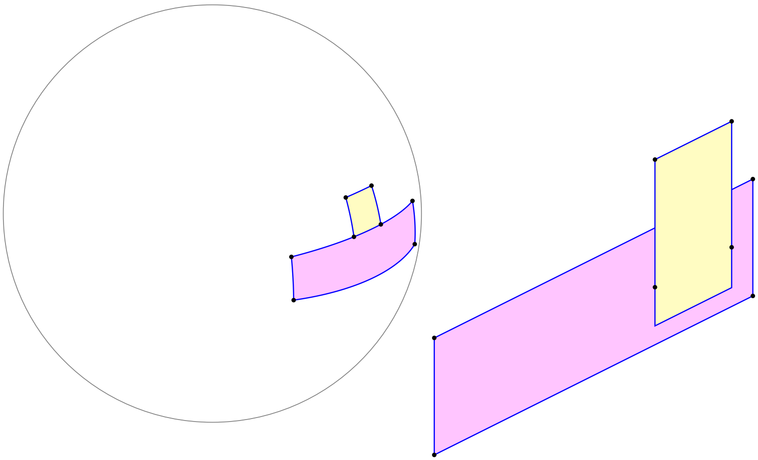
These observations justify the study of *edge guards*.



Problem 1. How many edge guards are needed for a polyhedron?

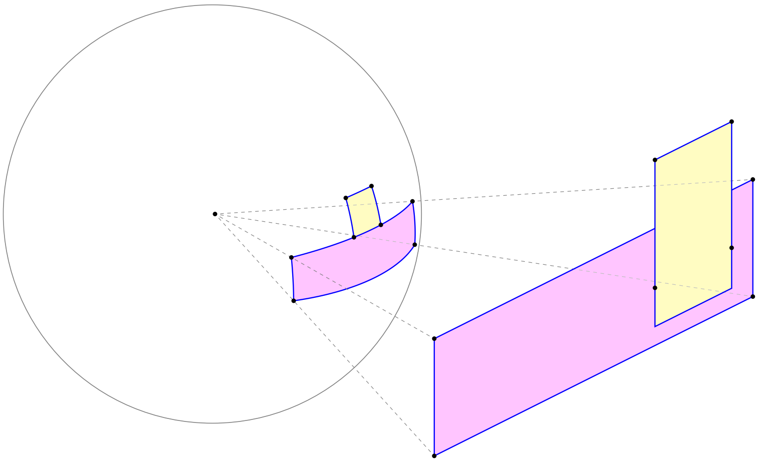
Problem 2. Assuming that there is a point guard on every vertex of a polyhedron, how many additional edge guards are needed?

Spherical Occlusion Diagrams: Introduction



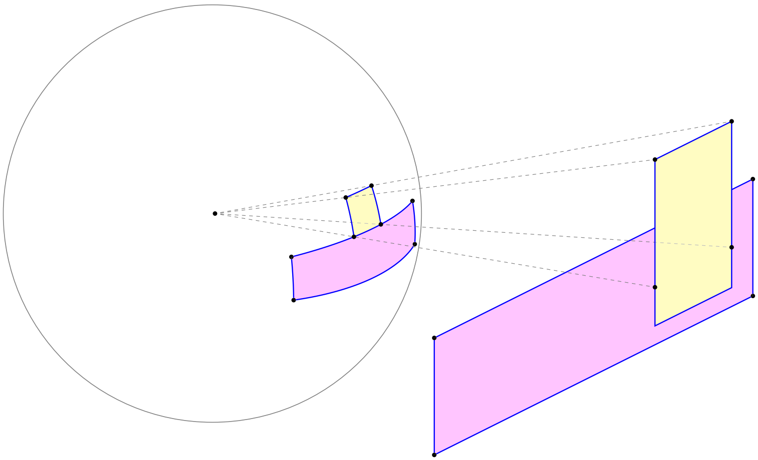
When polygons in $\mathbb{R}^3$ are orthographically projected onto a sphere, their edges become arcs of great circles.

Spherical Occlusion Diagrams: Introduction

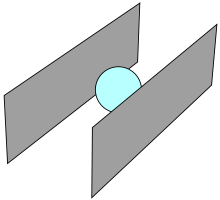


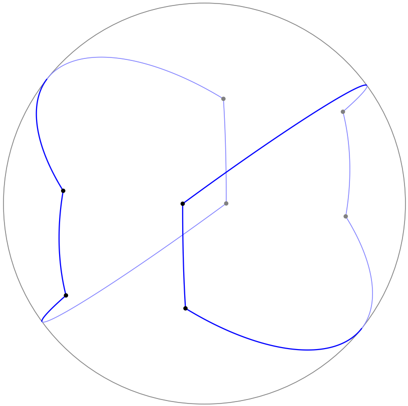
Moreover, when a polygon is partially hidden (i.e., **“occluded”**) by another, in the projection there are arcs feeding into other arcs.

Spherical Occlusion Diagrams: Introduction

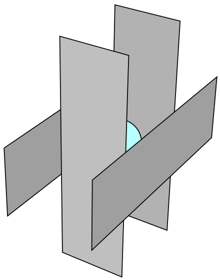


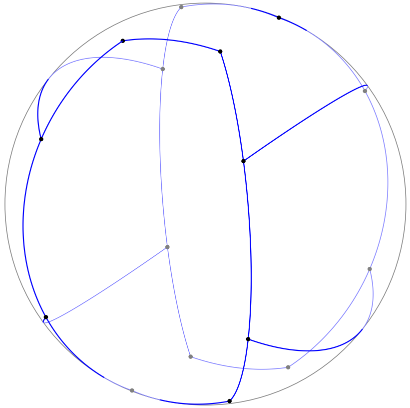
Moreover, when a polygon is partially hidden (i.e., **“occluded”**) by another, in the projection there are arcs feeding into other arcs.





If in an arrangement of polygons all vertices are occluded, then their edges project into a **“Spherical Occlusion Diagram”**.





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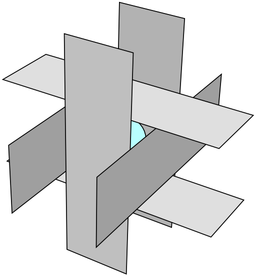
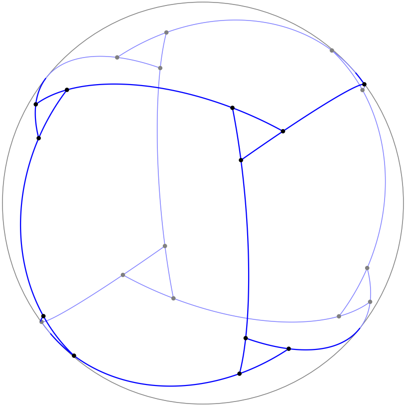
Spherical Occlusion Diagrams: Introduction





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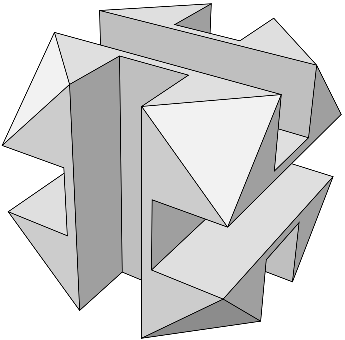
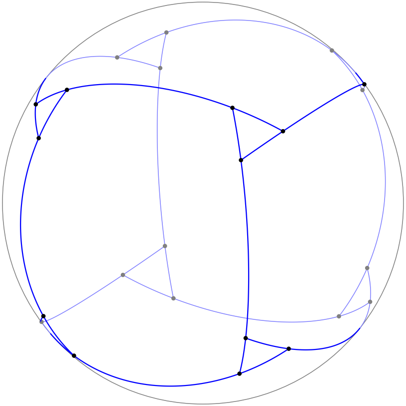
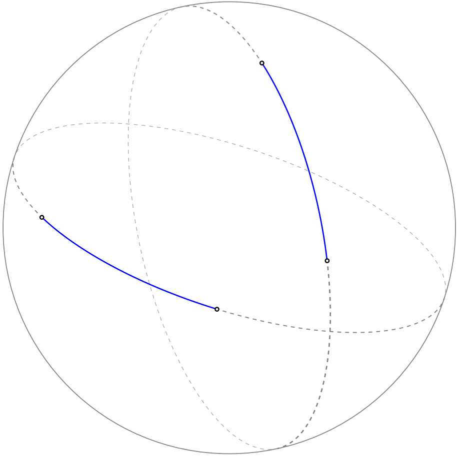




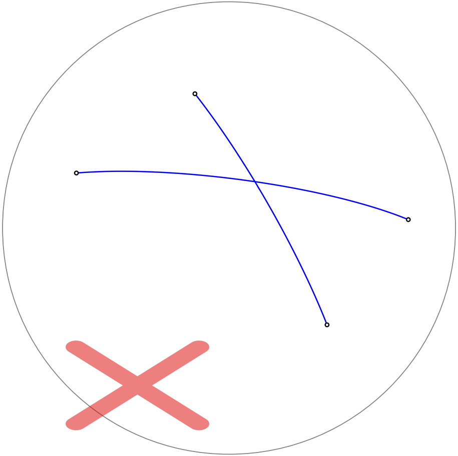
In particular, this applies to polyhedra: if all vertices are occluded, then the 1-skeleton projects into a Spherical Occlusion Diagram.

Spherical Occlusion Diagrams: Definition



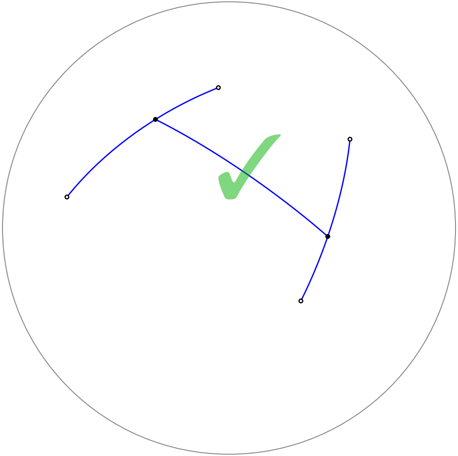
A **Spherical Occlusion Diagram**, or just “Diagram”, is a finite non-empty collection of arcs of great circles on the unit sphere.

Spherical Occlusion Diagrams: Definition



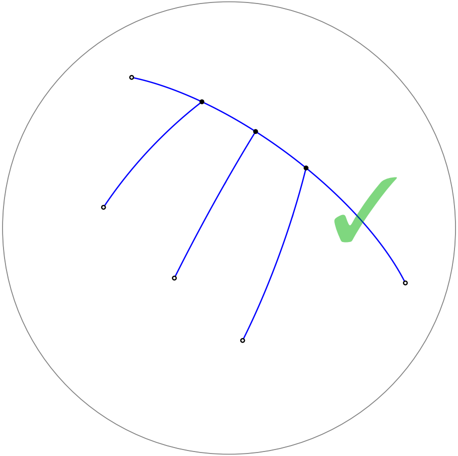
All arcs in a Diagram must be internally disjoint.

Spherical Occlusion Diagrams: Definition



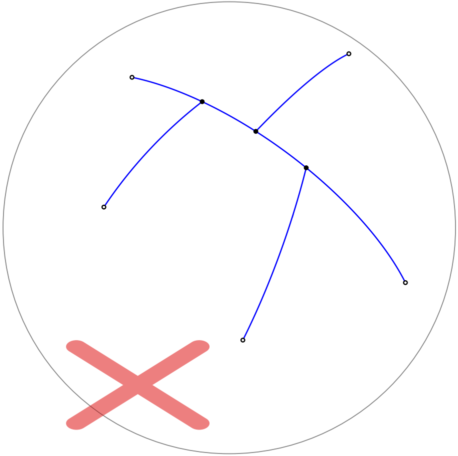
Both endpoints of each arc in a Diagram must lie in the interiors of some other arcs in the Diagram (every arc **"feeds into"** two arcs).

Spherical Occlusion Diagrams: Definition



All the arcs in a Diagram that feed into the same arc must reach it from the same side.

Spherical Occlusion Diagrams: Definition



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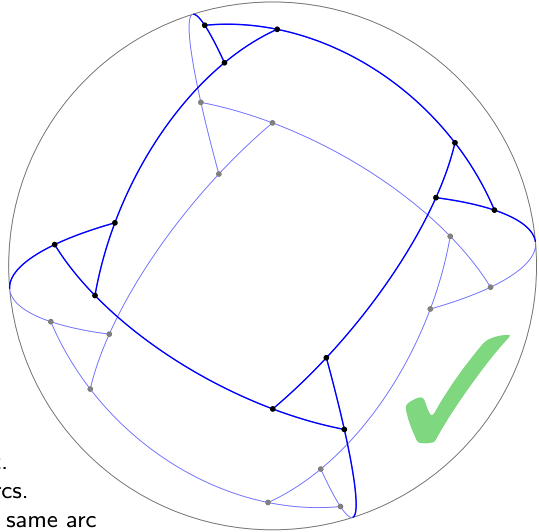
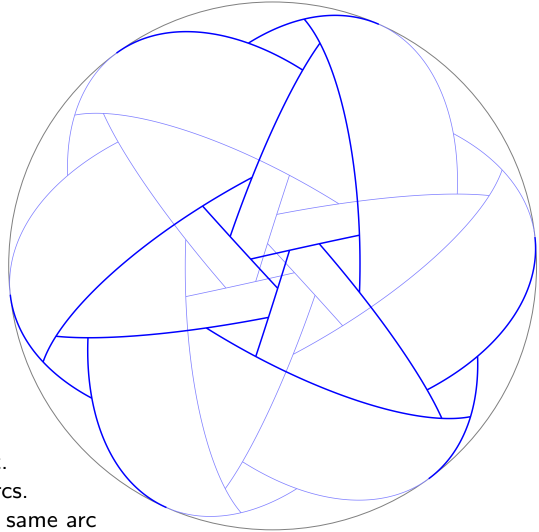


Diagram axioms:

1. Arcs are internally disjoint.
2. Each arc feeds into two arcs.
3. All arcs that feed into the same arc reach it from the same side.



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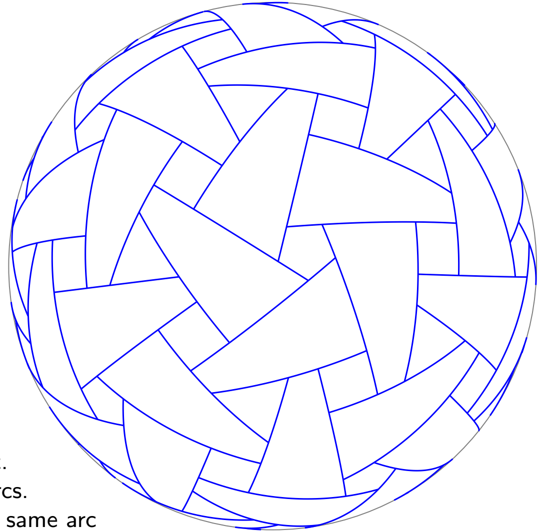

Diagram axioms:

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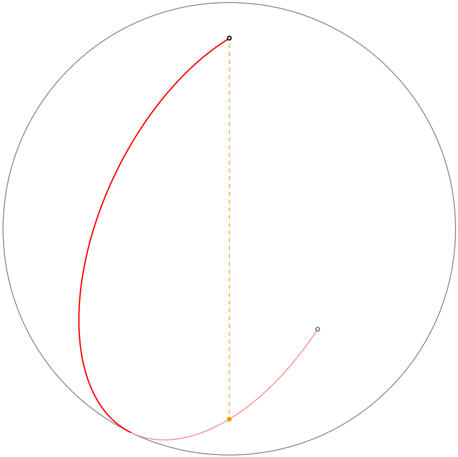
Spherical Occlusion Diagrams: Basic properties

Proposition

No arc in a Diagram is longer than a great semicircle.



Proof. Otherwise it would have arcs feeding into it from both sides.

Spherical Occlusion Diagrams: Basic properties

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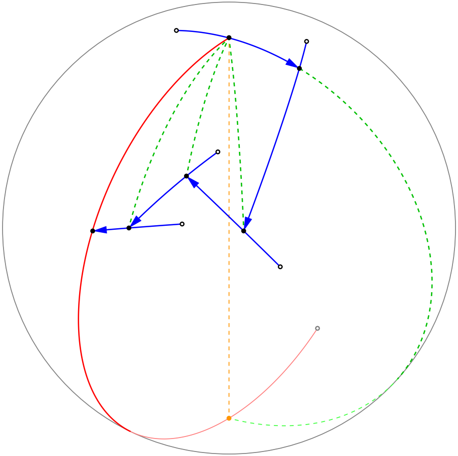

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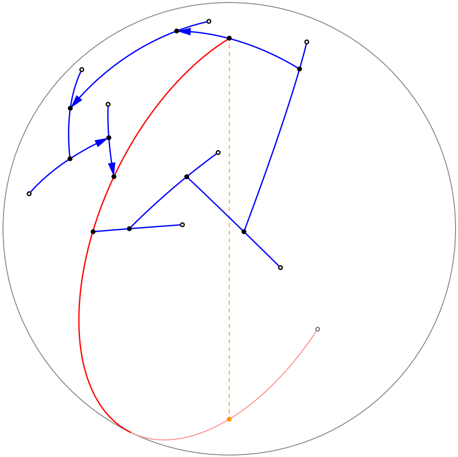
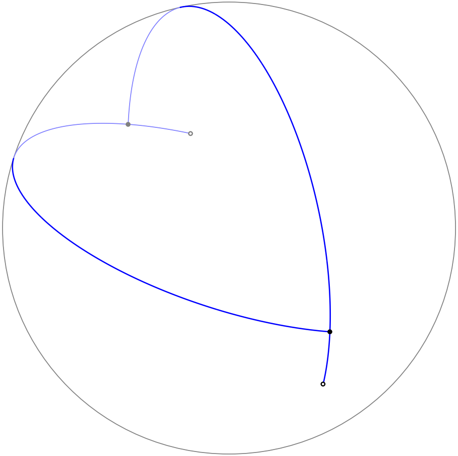


Proof. Otherwise it would have arcs feeding into it from both sides.

Corollary

No two arcs in a Diagram feed into each other.

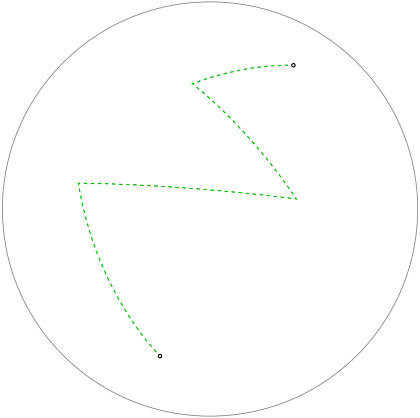


Proof. Otherwise they would be longer than a great semicircle.

Spherical Occlusion Diagrams: Basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

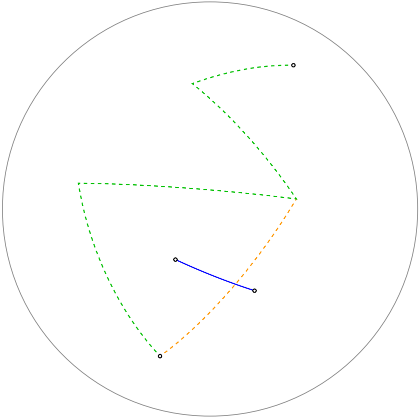


Proof. Two points in the same region can be connected by a chain of arcs of great circles that does not intersect the Diagram.

Spherical Occlusion Diagrams: Basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

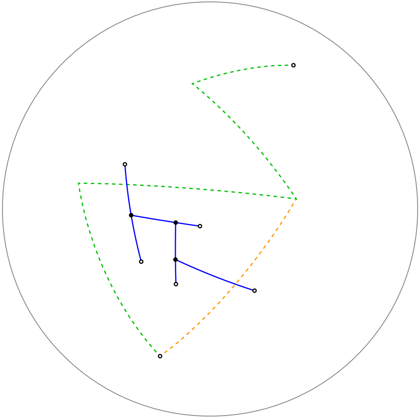


The arc joining the first and the third vertex of the chain do not intersect the Diagram, either...

Spherical Occlusion Diagrams: Basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

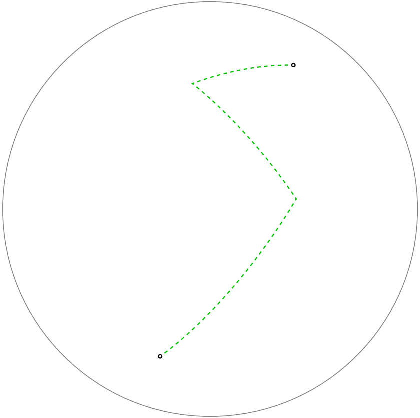


...Otherwise, following the Diagram we would intersect the first two arcs in the chain, which is impossible by assumption.

Spherical Occlusion Diagrams: Basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

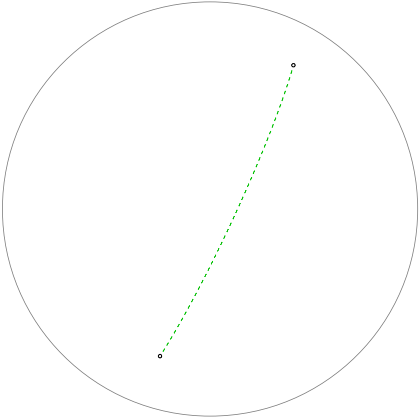


So we can simplify the chain, reducing it by one arc. Inductively repeating this reasoning, we can reduce the chain to a single arc.

Spherical Occlusion Diagrams: Basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

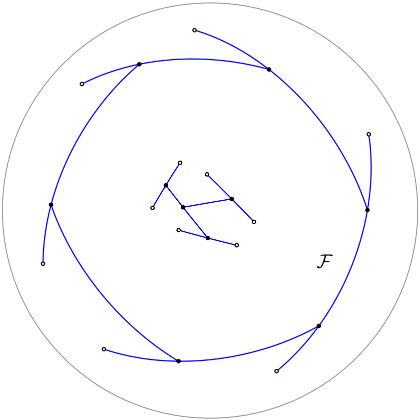


Any two points in the region are connected by an arc of a great circle that does not intersect the Diagram; hence, it is convex.

Spherical Occlusion Diagrams: Basic properties

Corollary

Every Diagram is connected.

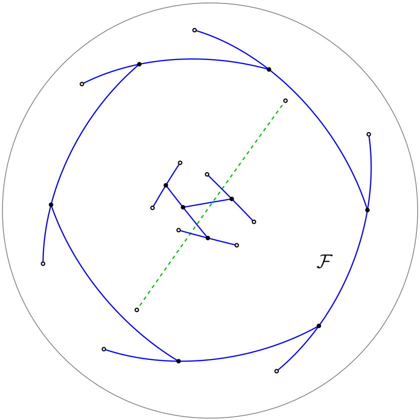


Proof. If there are two connected components, each of them is a Diagram. So, one is contained in a tile $\mathcal{F}$ determined by the other.

Spherical Occlusion Diagrams: Basic properties

Corollary

Every Diagram is connected.

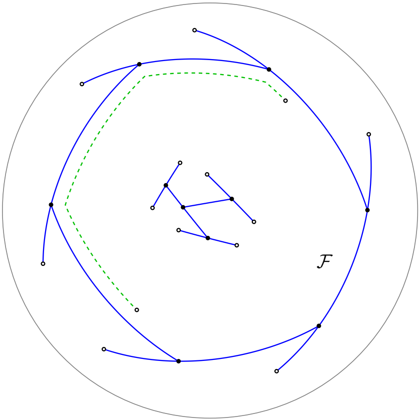


Take an arc in $\mathcal{F}$ with endpoints close to the first component that intersects the second component.

Spherical Occlusion Diagrams: Basic properties

Corollary

Every Diagram is connected.



The arc can be replaced by a chain that intersects neither connected component of the Diagram.

Spherical Occlusion Diagrams: Basic properties

Corollary

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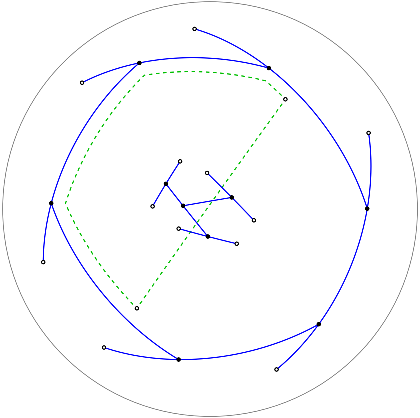
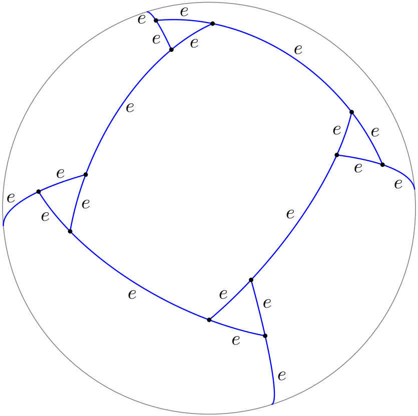


So its endpoints are in the same tile determined by the whole Diagram, and this tile cannot be convex.

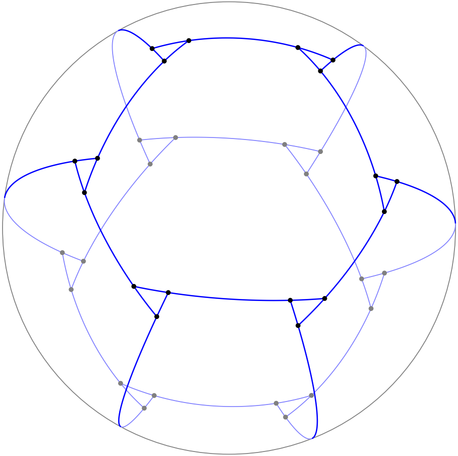
Spherical Occlusion Diagrams: Basic properties

Proposition

A Diagram with n arcs partitions the sphere into $n + 2$ tiles.

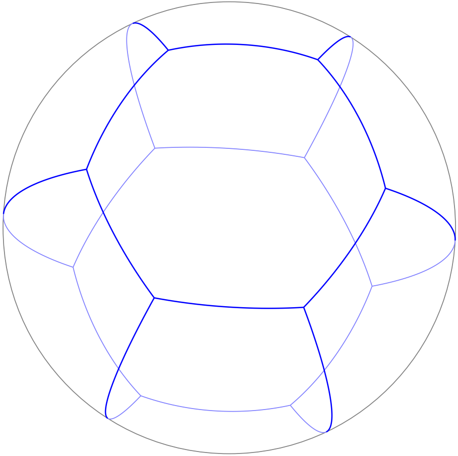


Proof. A Diagram induces a planar graph with v vertices and $n + v$ edges. By Euler's formula, $f + v = n + v + 2$, hence $f = n + 2$.



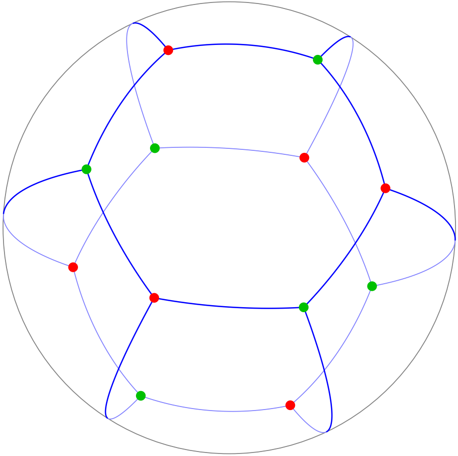
How can we automatically generate large classes of Diagrams?

Spherical Occlusion Diagrams: Construction

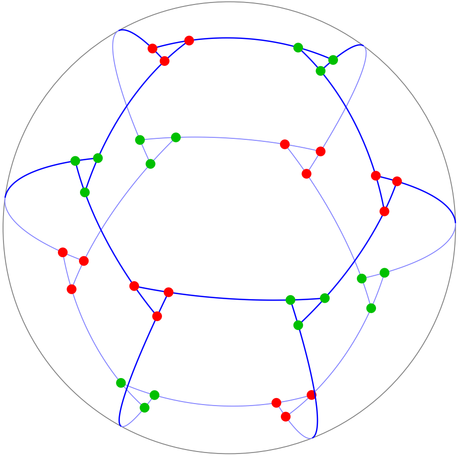


Start from a subdivision of the sphere into strictly convex tiles, where each tile has an even number of edges.

Spherical Occlusion Diagrams: Construction

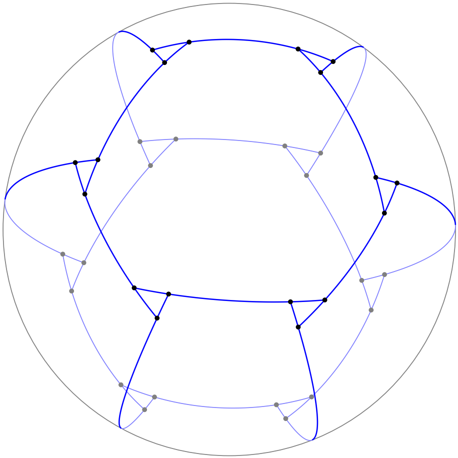


Note that the 1-skeleton of the tiling is <u>bipartite</u>, because it has no odd cycles.



We can turn each vertex of the tiling into a **“swirl”** going clockwise or counterclockwise according to the bipartition of the 1-skeleton.

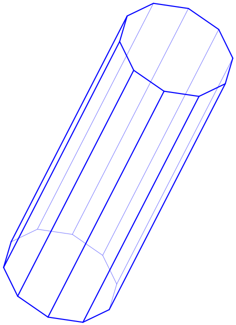
Spherical Occlusion Diagrams: Construction

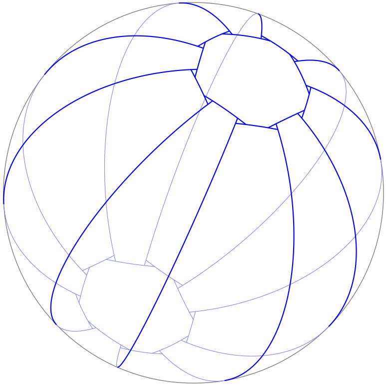


This operation defines a natural correspondence between even-sided spherical tilings and so-called *swirling Diagrams*.

Swirling Diagrams: Examples

This method enables the automatic construction of swirling Diagrams from convex tilings of the sphere or convex polyhedra.





Prisms with even-sided bases

Swirling Diagrams: Examples

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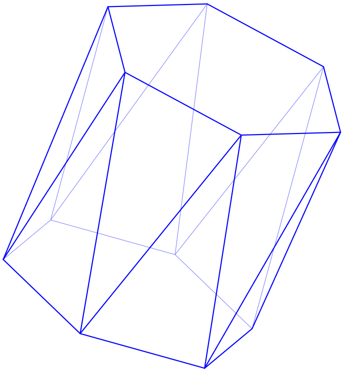

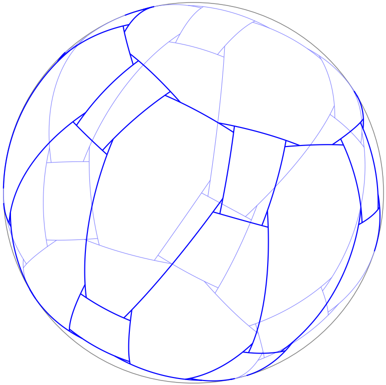

Truncated antiprisms

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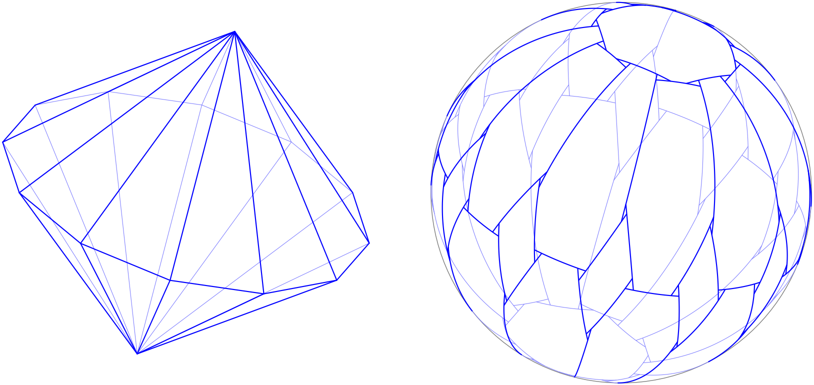

Truncated bipyramids with even-degree vertices

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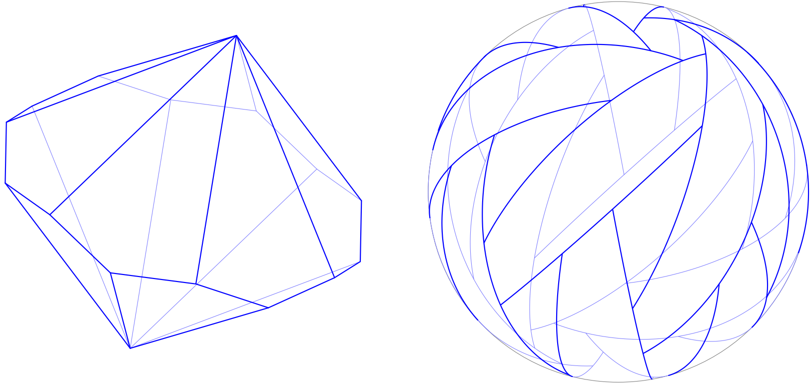

Trapezohedra

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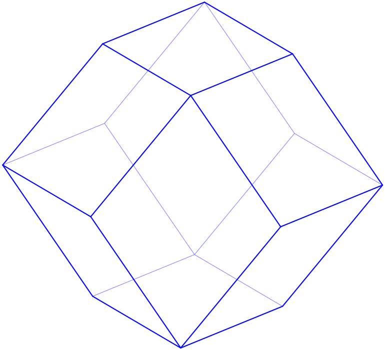

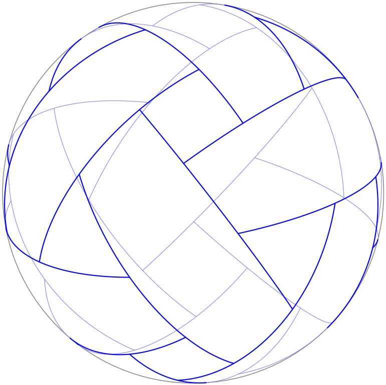

Rhombic dodecahedron

Swirling Diagrams: Examples

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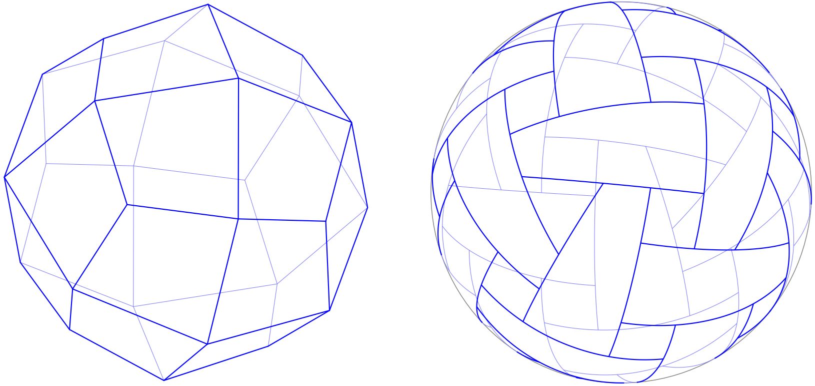

Deltoidal icositetrahedron

Swirling Diagrams: Examples

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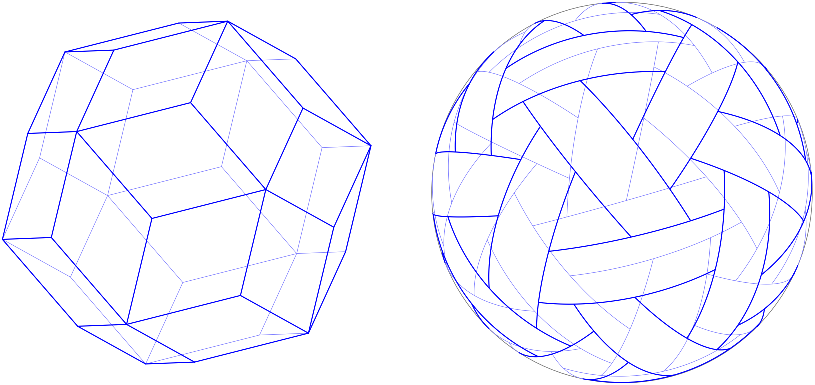

Rhombic triancontahedron

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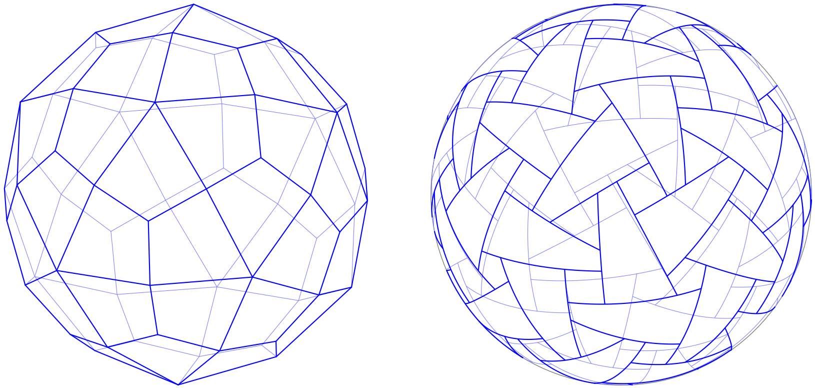

Deltoidal hexecontahedron

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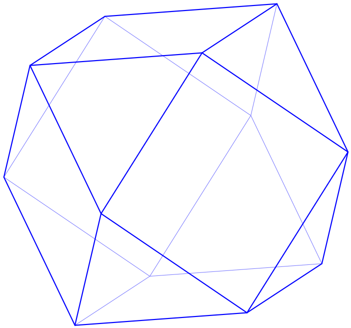
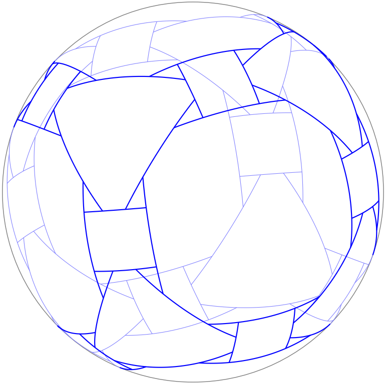
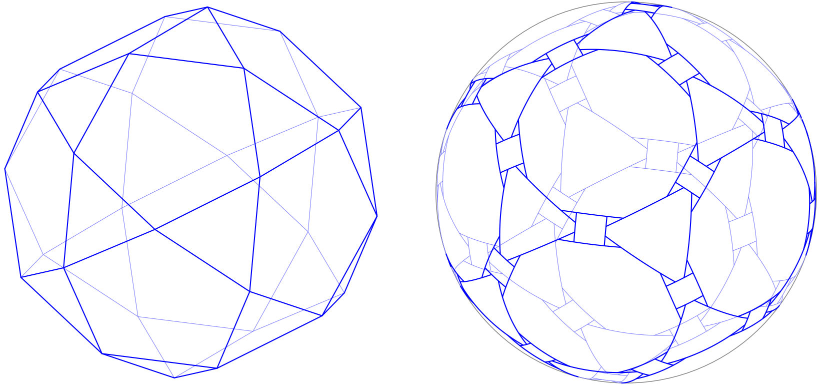




Truncated cuboctahedron

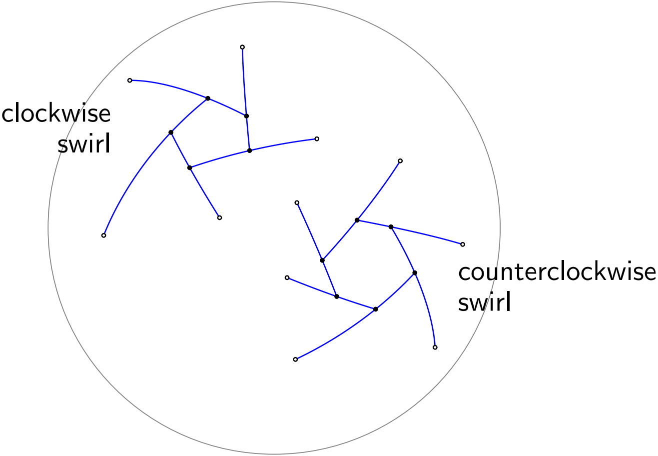
Swirling Diagrams: Examples

This method enables the automatic construction of swirling Diagrams from convex tilings of the sphere or convex polyhedra.



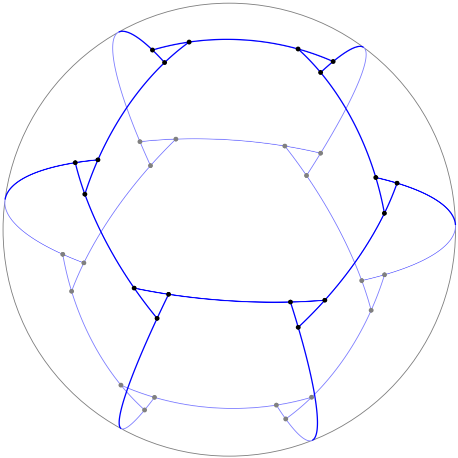
Truncated icosidodecahedron

Swirling Diagrams: Alternative definitions



As we saw, a **swirl** in a Diagram is a cycle of arcs such that each arc feeds into the next going clockwise or counterclockwise.

Swirling Diagrams: Alternative definitions



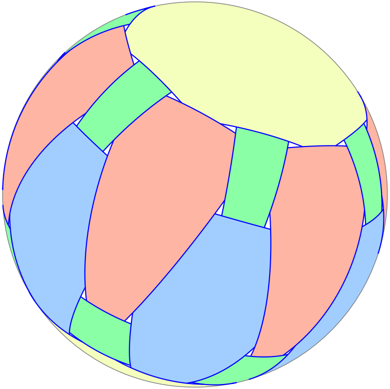
Observation

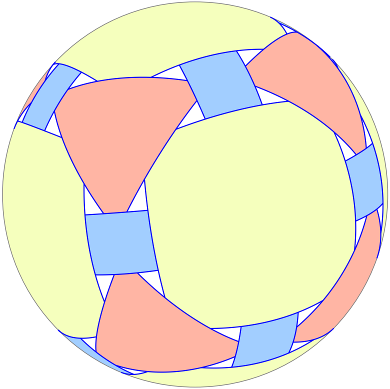
A Diagram is **swirling** *if and only if every arc is part of two swirls.*

Swirling Diagrams: Alternative definitions

Observation

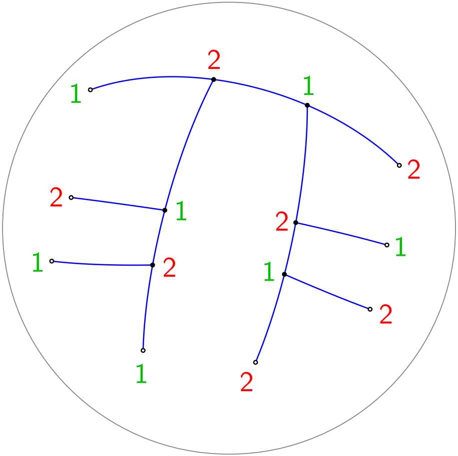
If in an arrangement of polygons all vertices are occluded, and each edge occludes vertices of at most one polygon, then the edges project into a swirling Diagram.





Uniform Diagrams

Each arc in a Diagram feeds into exactly two arcs. So, the average number of arcs feeding into a given arc of a Diagram is two.

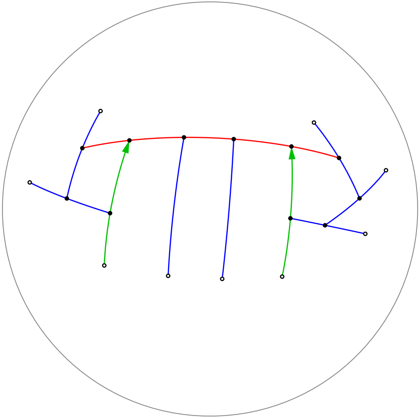


A Diagram is said **uniform** if each arc has two arcs feeding into it.

Uniform Diagrams

Proposition

All <u>swirling</u> Diagrams are <u>uniform</u>.

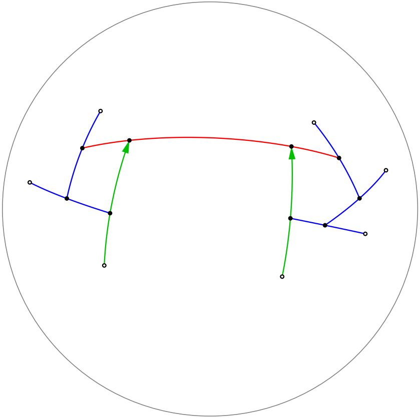


Proof. In a swirling Diagram, each arc is part of two distinct swirls, and so <u>at least two arcs</u> feed into it.

Uniform Diagrams

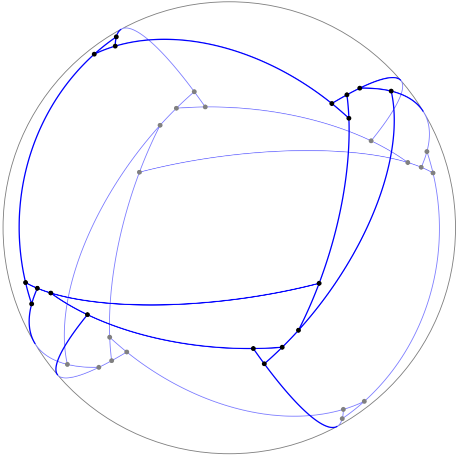
Proposition

All swirling Diagrams are uniform.



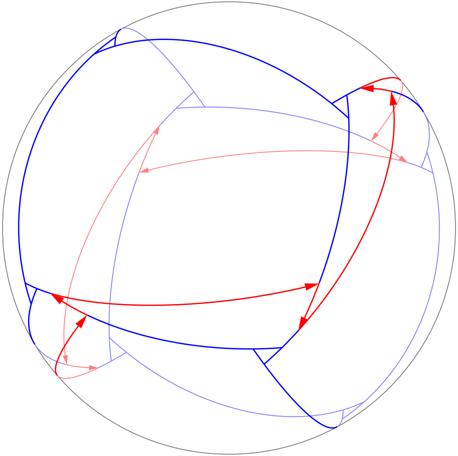
But each arc has two arcs feeding into it on average, so it must have exactly two arcs feeding into it.

Uniform Diagrams



The converse is not true: there are uniform Diagrams that are not swirling.

Uniform Diagrams

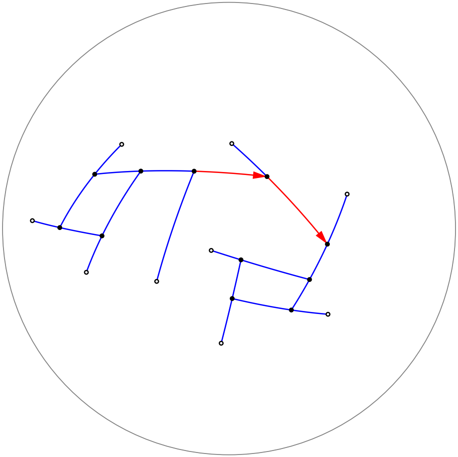


Note that the (portions of) arcs that are not part of a swirl form a <u>cycle</u> where each arc feeds into the next: *this is not a coincidence...*

Uniform Diagrams

Proposition

In a uniform Diagram, the non-swirling arcs form <u>*disjoint cycles*</u>*.*

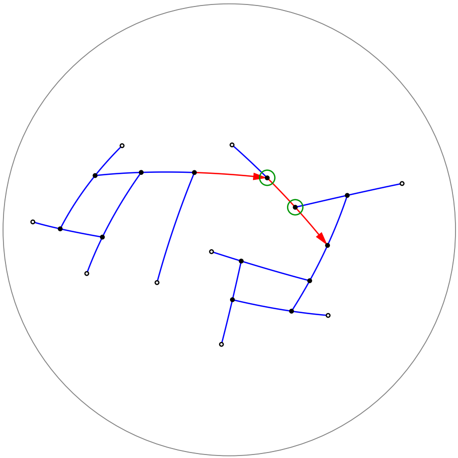


Proof. Consider the last arc in a chain of non-swirling arcs.

Uniform Diagrams

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.

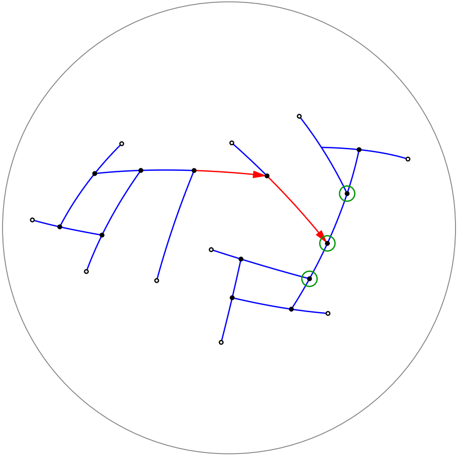


This arc cannot form a swirl with the arc it feeds into (axiom 3).

Uniform Diagrams

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.

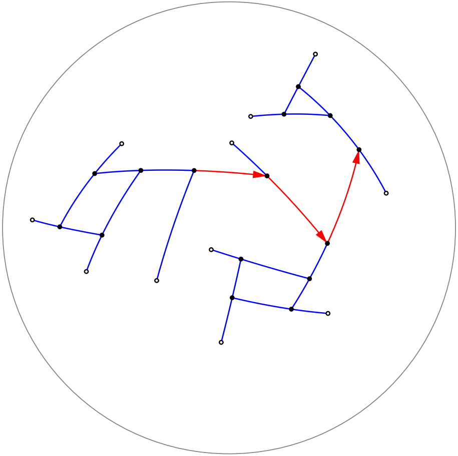


So, the arc it feeds into cannot be part of two swirls (uniformity).

Uniform Diagrams

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.

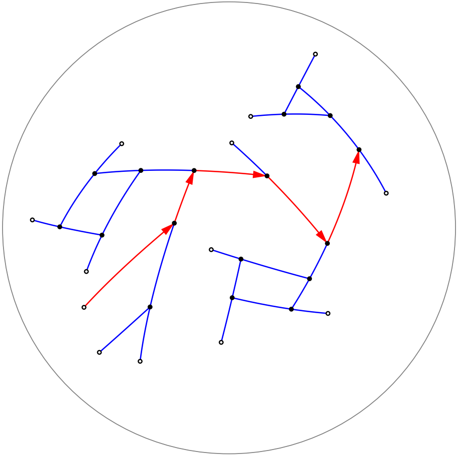


Therefore, the chain must be followed by another non-swirling arc.

Uniform Diagrams

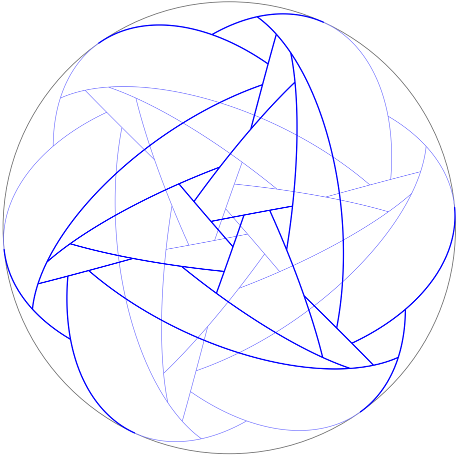
Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.



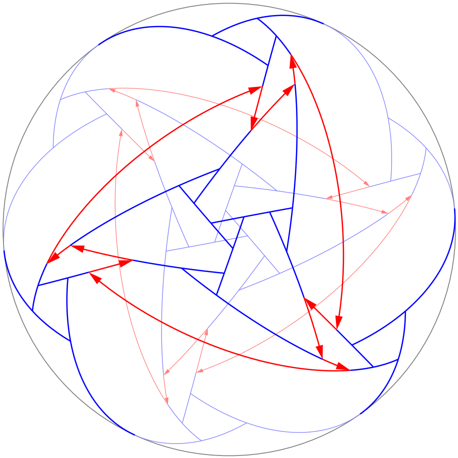
Moreover, the chain can be uniquely extended backwards.

Uniform Diagrams



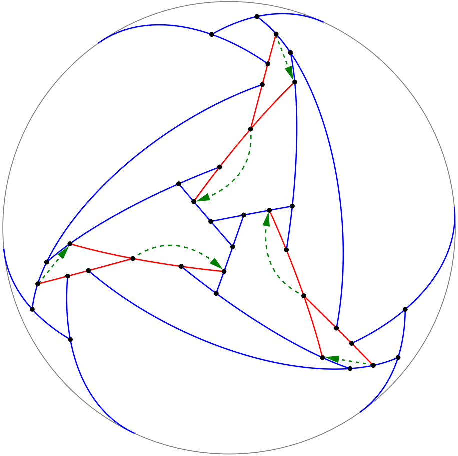
Uniform Diagrams can have any number of unboundedly long cycles of non-swirling arcs.

Uniform Diagrams



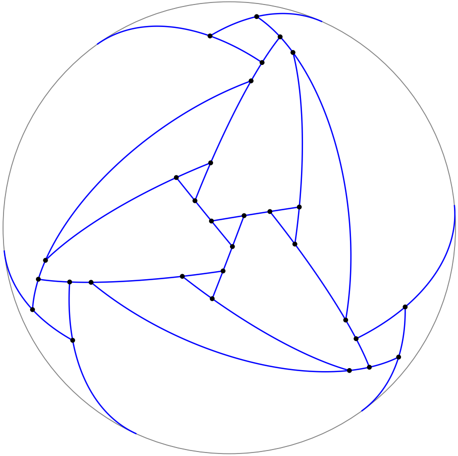
Uniform Diagrams can have any number of unboundedly long cycles of non-swirling arcs.

Uniform Diagrams



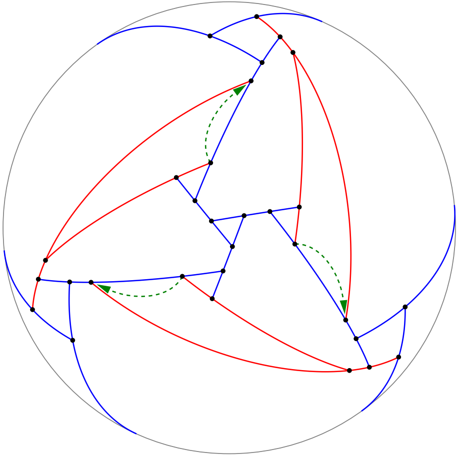
By suitably merging consecutive arcs in each cycle, we can transform any uniform Diagram into a swirling one.

Uniform Diagrams



By suitably merging consecutive arcs in each cycle, we can transform any uniform Diagram into a swirling one.

Uniform Diagrams



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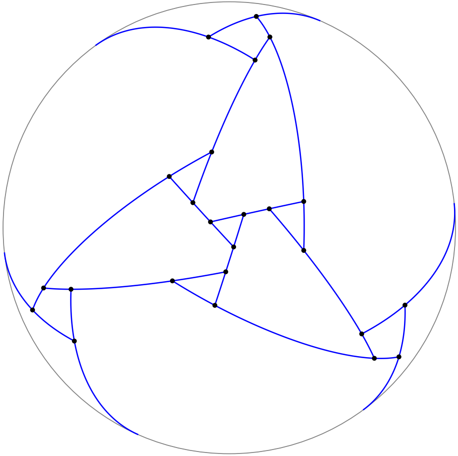
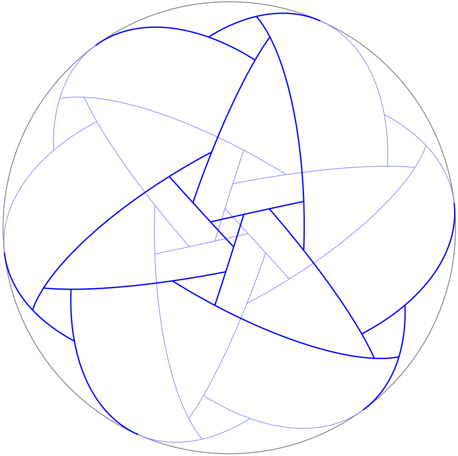
Uniform Diagrams



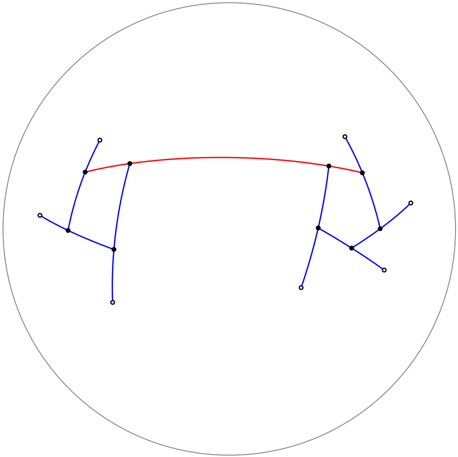
By suitably merging consecutive arcs in each cycle, we can transform any uniform Diagram into a swirling one.

Uniform Diagrams



By suitably merging consecutive arcs in each cycle, we can transform any uniform Diagram into a swirling one.

Swirl Graph

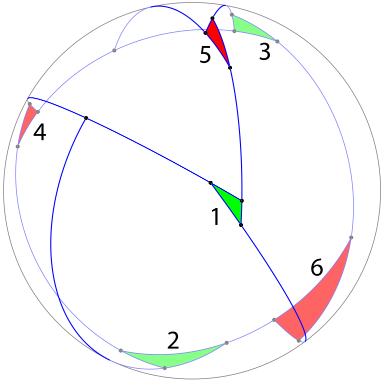


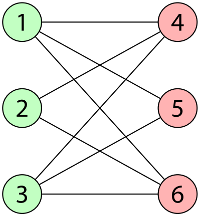
The **Swirl Graph** of a Diagram is an undirected multigraph on the set of swirls. For each arc shared by two swirls, there is an edge in the Swirl Graph.

Swirl Graph

Theorem

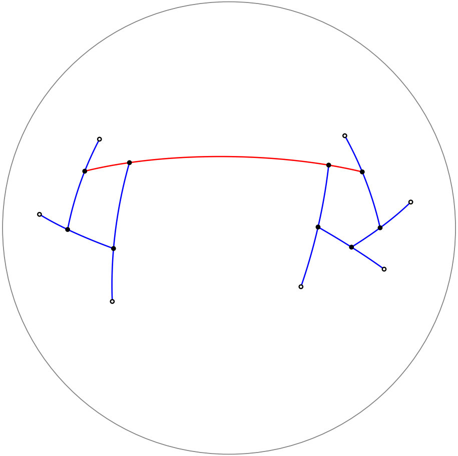
The Swirl Graph of a Diagram is a simple planar bipartite graph with non-empty partite sets.





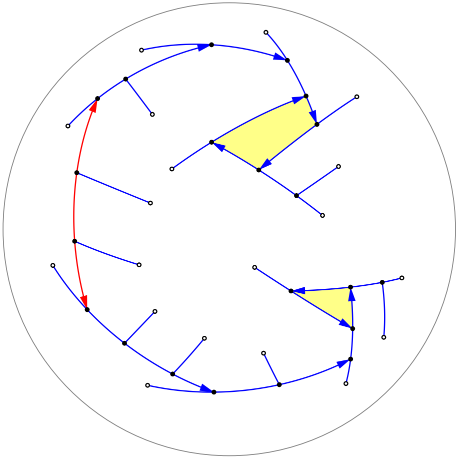
Proof. Obviously the Swirl Graph is spherical, hence planar. The bipartition is given by the clockwise and counterclockwise swirls...

Swirl Graph



Each edge in the Swirl Graph must connect a clockwise and a counterclockwise swirl. So the Swirl Graph is <u>bipartite</u>.

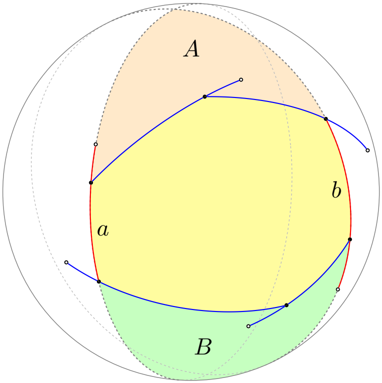
Swirl Graph



To find a (counter)clockwise swirl, start anywhere and follow the Diagram (counter)clockwise. Hence the partite sets are not empty.

Swirl Graph

Assume that the yellow swirl shares arcs a and b with another swirl. The second swirl must be located in the highlighted spherical lune.

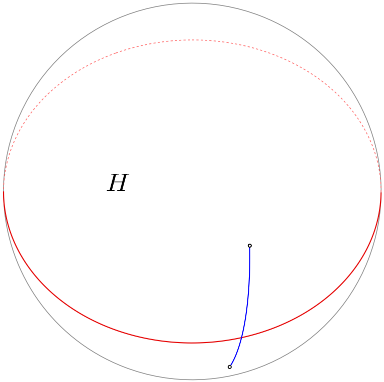


Since a goes upward, the second swirl must be in A. But b goes downward, so the second swirl must be in B: contradiction. Hence, the Swirl Graph is <u>simple</u>.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

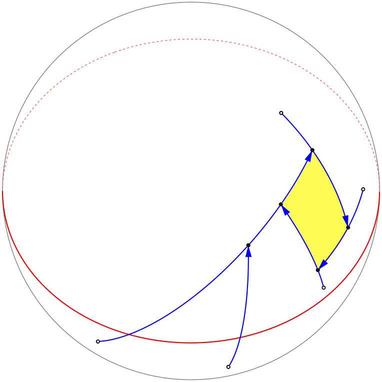


Proof. Take any hemisphere H. Since the Diagram is connected and tiles are convex, there is an arc crossing the boundary of H.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

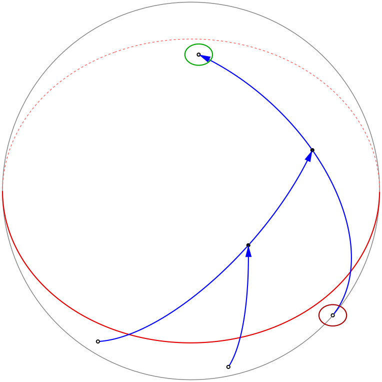


Follow the Diagram clockwise starting from this arc. If we remain in H, we eventually find a clockwise swirl fully contained in H.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

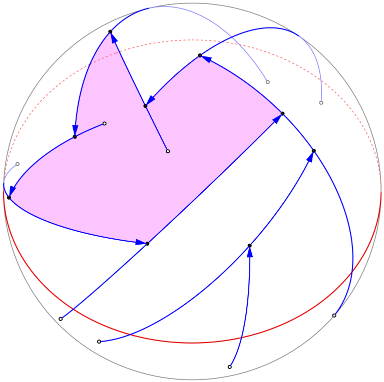


Otherwise, we find one arc whose clockwise endpoint is outside H. But then, the other endpoint is in H. Reach that endpoint.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

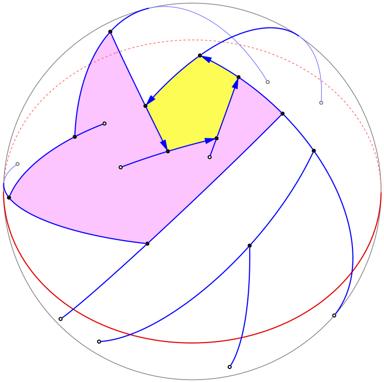


Continuing in this fashion, we either find a clockwise swirl in H, or eventually we enclose a region within H by going counterclockwise.

Hemisphere lemma

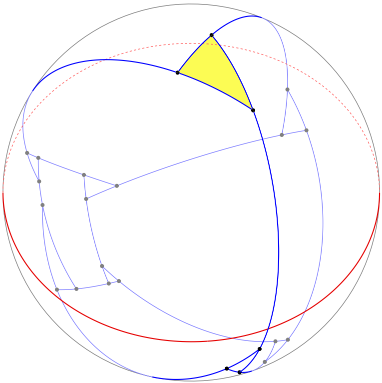
Lemma

In a Diagram, every hemisphere contains at least one full swirl.



In this case, starting from the boundary of the enclosed region and following the Diagram counterclockwise, we eventually find a swirl.

Hemisphere lemma

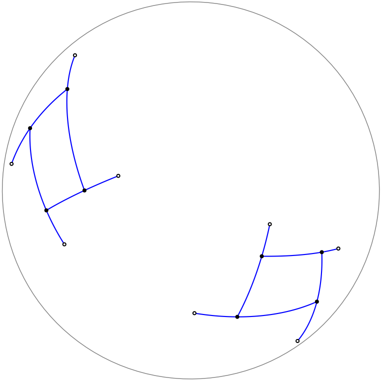


Note: In some cases, a hemisphere may contain exactly one swirl.

More swirls

Corollary

Every Diagram has at least 4 swirls.

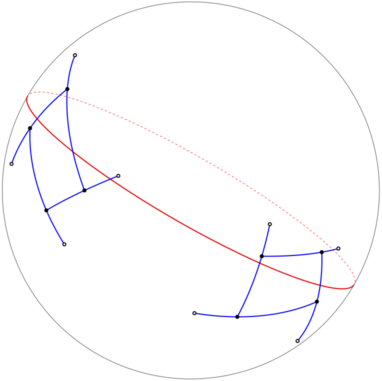


Proof. We already know that a Diagram has 2 swirls.

More swirls

Corollary

Every Diagram has at least 4 swirls.

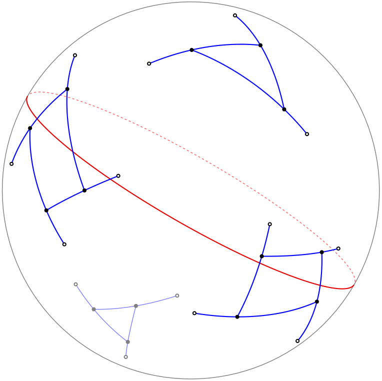


Take a great circle that properly intersects both swirls.

More swirls

Corollary

Every Diagram has at least 4 swirls.

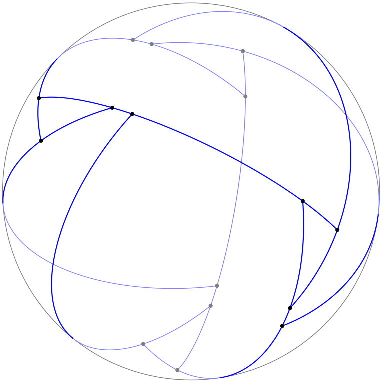


By the previous lemma, each hemisphere contains one new swirl.

Minimizing arcs (and swirls)

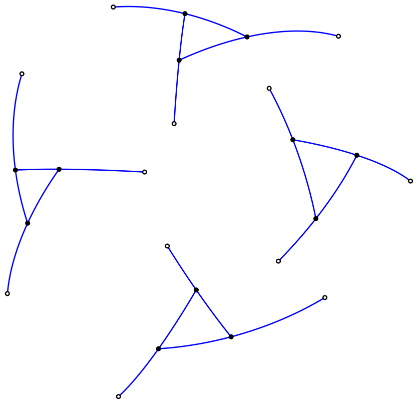
Theorem

Every Diagram has at least 8 arcs, and there exist Diagrams with exactly 8 arcs (and exactly 4 swirls).



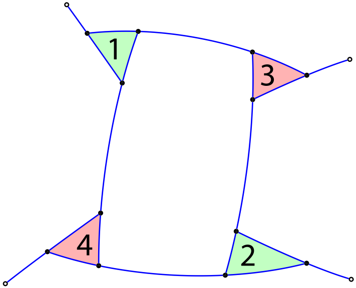
Proof. This is an example of a Diagram with 8 arcs and 4 swirls...

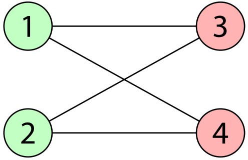
Minimizing arcs (and swirls)



We know that a Diagram has at least 4 swirls. Obviously, if they do not share any arcs, then the Diagram has at least 12 arcs.

Minimizing arcs (and swirls)



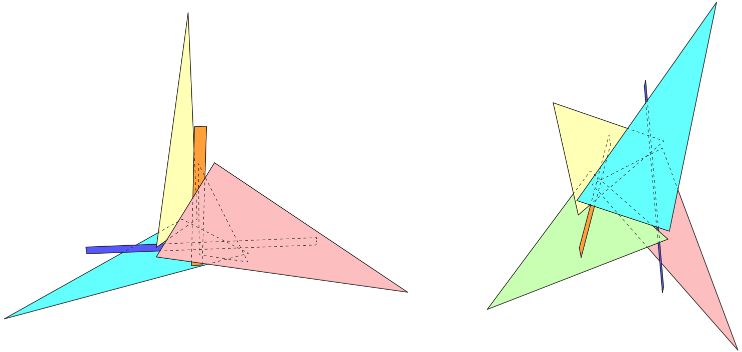


Since the Swirl Graph must be simple and bipartite, these 4 swirls can share at most 4 arcs. Thus the Diagram has at least 8 arcs.

Minimizing visible edges

Corollary

If a point does not see any vertices of a polyhedron, it sees at least 8 distinct edges. The bound is tight.



A lower-bound example can be constructed from this arrangement of 6 polygons, where the central point does not see any vertices and sees exactly 8 edges.

Summary

- **Spherical Occlusion Diagrams** occur naturally when studying points that see no vertices of a polyhedron.
- There is a straightforward correspondence between **swirling Diagrams** and even-sided convex tilings of the sphere.
- **Uniform Diagrams** can be obtained by augmenting swirling Diagrams with disjoint non-swirling cycles.
- Swirls are patterns frequently appearing in Diagrams. By studying **swirl graphs**, we obtain a new Art Gallery theorem: *If we see no vertices of a polyhedron, then we see 8+ edges.*

Future work

Conjecture

There are no swirling Diagrams with 8, 9, 10, 11, 13, 14, 15, 17, 21, 22, 23, or 29 arcs.

Conjecture

Every Diagram has at least 2 clockwise and 2 counterclockwise swirls.

Conjecture

Not every Diagram is the projection of a polyhedron's 1-skeleton.

Conjecture

Not every Diagram is combinatorially equivalent to the projection of a polyhedron's 1-skeleton.

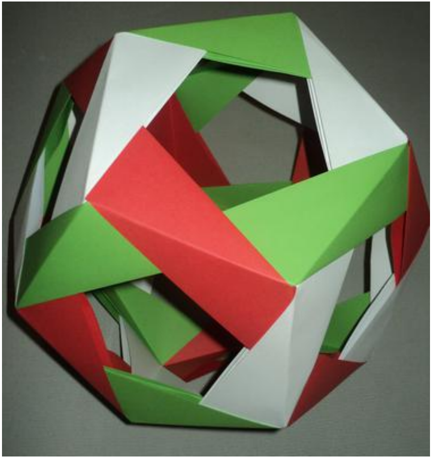
Conjecture

If a point does not see any vertices of a polyhedron, it sees at least 8 distinct faces.

Spherical Occlusion Diagrams in everyday life

Modular origami: kusudama

Spherical Occlusion Diagrams in everyday life



Modular origami: penultimate dodecahedron

Spherical Occlusion Diagrams in everyday life

Modular origami: penultimate truncated icosahedron

Spherical Occlusion Diagrams in everyday life

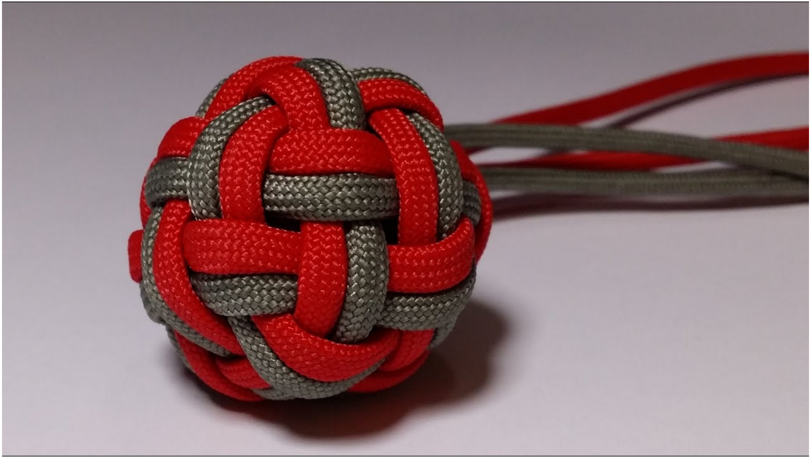
Kirigami ball decoration

Spherical Occlusion Diagrams in everyday life

Monkey's fist knot

Single-thread globe knot

Spherical Occlusion Diagrams in everyday life

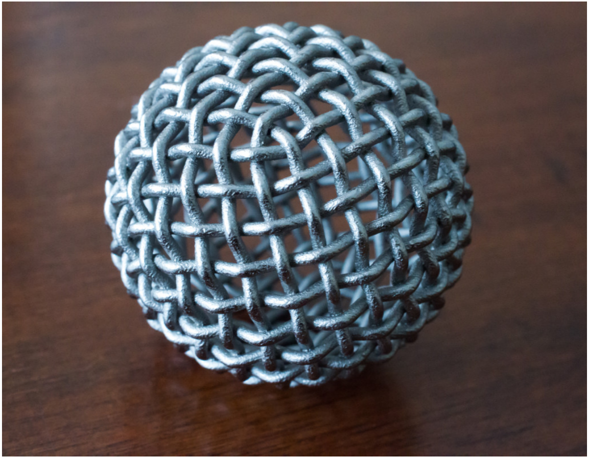


Double-thread globe knot

Spherical Occlusion Diagrams in everyday life

Herringbone pineapple knot

Spherical Occlusion Diagrams in everyday life



Stainless-steel globe knot

Spherical Occlusion Diagrams in everyday life

Sepak-takraw ball

Spherical Occlusion Diagrams in everyday life

Rattan balls

Spherical Occlusion Diagrams in everyday life

Rattan vase

Spherical Occlusion Diagrams in everyday life

Toroidal Occlusion Diagrams...?