Compaction Puzzles and Games

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Joint work with Hugo Akitaya, Maarten Löffler, and Anika Rounds

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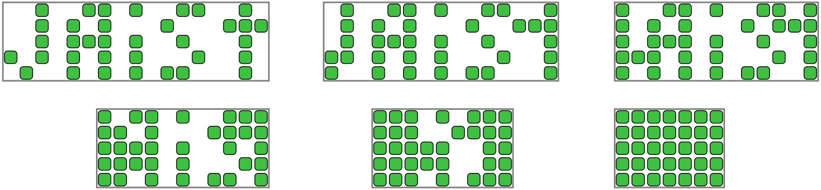
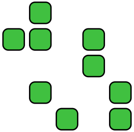


Overview

- Compaction puzzle (1 player)
  - Forming an $a \times b$ box is NP-complete
  - Forming an $a \times 2$ box is easy
- Compaction games (2 players)
  - Last-Move-Wins
  - k-in-a-Row
  - Hungry-Tokens
  - Others...

Compaction puzzle

There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



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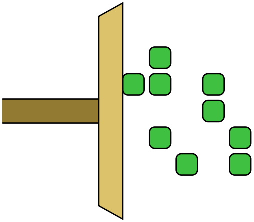

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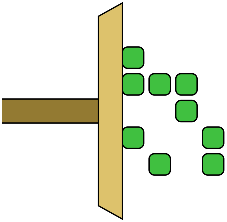

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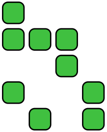

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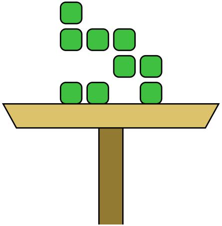

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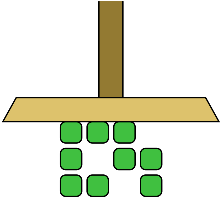

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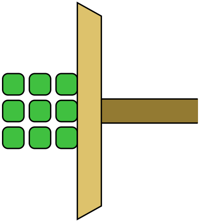

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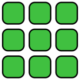
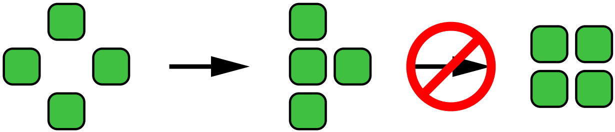


Problem: Given n tokens, can we push them into an $a \times b$ box?

Counterexample

There are cases where n^2 tokens cannot be pushed into a square:

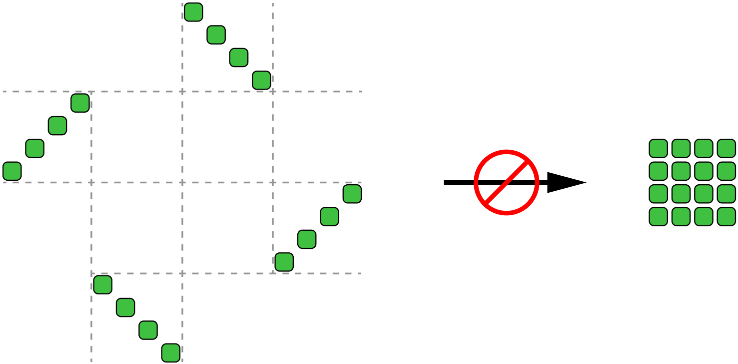


Observation

The maximum number of tokens in the same row (resp. column) can never decrease.

Sparse configurations

What if the configuration is *"sparse"*, i.e., no row or column has more than one token? Forming a square may still be impossible:

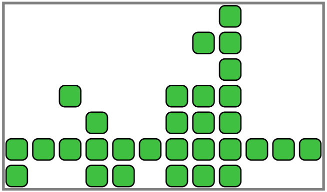


Open problem

Is there a sparse configuration of $n^2 = 9$ tokens that cannot be pushed into a square? (We have such examples for all $n > 3$.)

The compaction puzzle is in NP

Once there are a row and a column with no empty cells, the bounding box can no longer shrink. At this point, if the configuration is not a full box, it will never be a full box.



It is now easy to prove that solvable compaction puzzles have a polynomial-length solution. Hence, the compaction puzzle is in NP.

Exact-Hitting-Set

To prove that the compaction puzzle is NP-complete, we will give a reduction from the following NP-complete problem:

Exact-Hitting-Set

Input: A universe set $U = \{1, 2, \ldots, n\}$, m subsets $S_1, S_2, \ldots, S_m \subseteq U$, and an integer k.

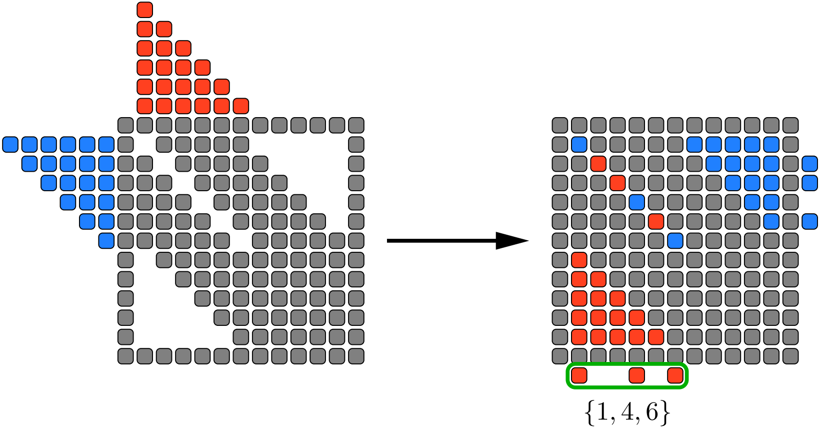
Output: YES if there is a subset $H \subseteq U$ of exactly k elements such that $|H \cap S_i| = 1$ for all $1 \leq i \leq m$; NO otherwise.

Example: $U = \{1, 2, 3, 4, 5, 6\}$,
$S_1 = \{1, 3, 6\}$, $S_2 = \{2, 3, 5\}$, $S_3 = \{4, 5\}$, $S_4 = \{3, 4, 6\}$,
$k = 3$.

A possible solution is $H = \{1, 2, 4\}$.

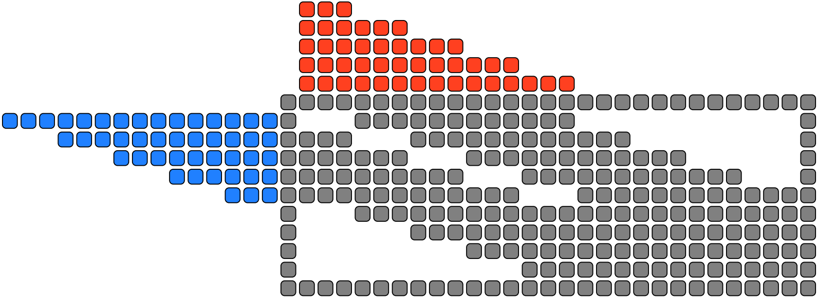
Selector gadget

The *selector gadget* allows us to construct any subset H of a universe set $U = \{1, 2, \ldots, n\}$. Here is an example with $n = 6$:



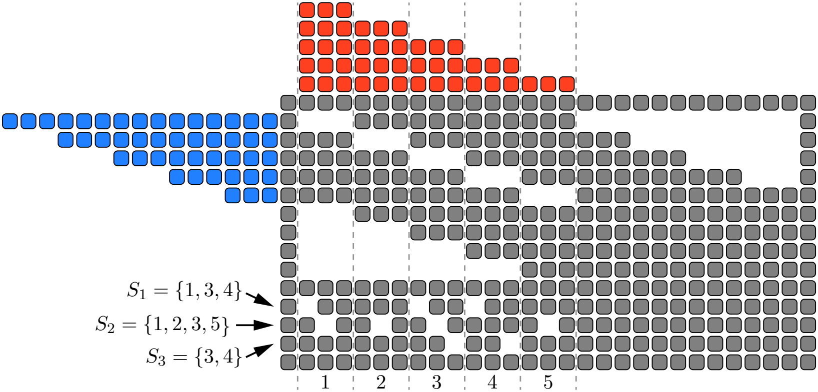
The compaction puzzle is NP-complete

Let us scale up the selector gadget horizontally by a factor m.



The compaction puzzle is NP-complete

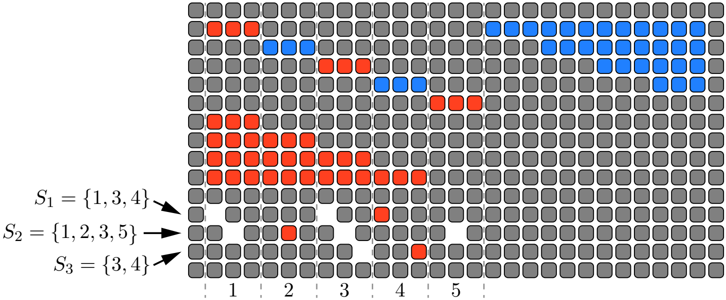
Next, we attach a section representing the sets $S_1, S_2, \ldots, S_m$.



The compaction puzzle is NP-complete

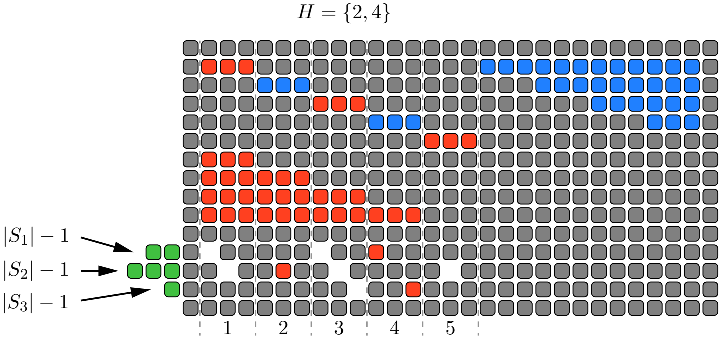
The selector gadget allows the player to construct any set $H \subseteq U$.

$H = \{2, 4\}$



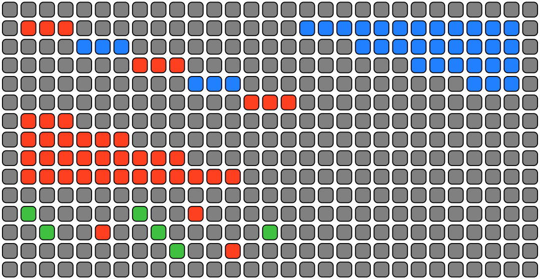
The compaction puzzle is NP-complete

We add some extra tokens to ensure that each S_i contains exactly one element from the set H selected by the player.



The compaction puzzle is NP-complete

The extra tokens fit if and only if the set H chosen by the player solves $\textsc{Exact-Hitting-Set}$.



The compaction puzzle is NP-complete

In the full construction, we add some extra structure to take care of the blue tokens that spill to the right of the selector gadget.

We also move a token from the top left to the bottom right to force the player to only push right or down.

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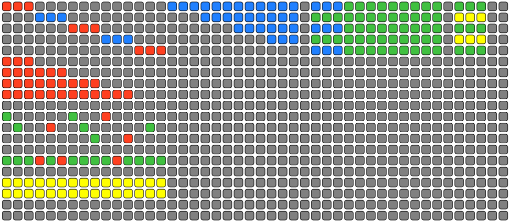


We also move a token from the top left to the bottom right to force the player to only push right or down.

Compaction puzzle: Results

We have proved:

Theorem (Akitaya et al., 2016)

Deciding if tokens can be pushed into an $a \times b$ box is NP-complete.

Corollary

Deciding if tokens can be pushed into a square is NP-complete.

More results from the same paper:

H. Akitaya, G. Aloupis, M. Löffler, and A. Rounds
"Trash compaction", in *Proceedings of EuroCG 2016*

Theorem

Given $2n$ tokens that occupy at most 3 rows, we can decide in $O(n)$ time if they can be pushed into an $n \times 2$ box.

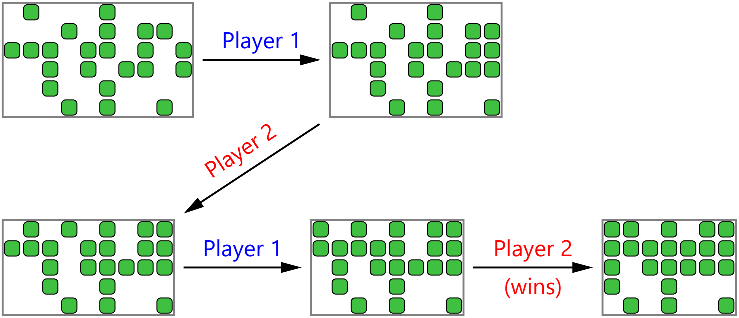
Theorem

Given $2n$ tokens that occupy $k > 3$ rows, we can decide in $O(n^{2(k-2)})$ time if they can be pushed into an $n \times 2$ box.

Compaction game: Last-Move-Wins

In a *compaction game*, two players take turns pushing tokens.

Rule 1: A push should always reduce the size of the bounding box.

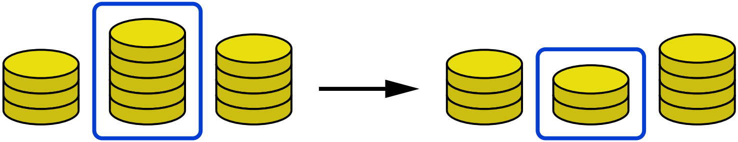


Rule 2: The player who cannot reduce the bounding box loses.

The game of Nim

Note that this compaction game is *impartial*: in any configuration, the set of legal moves does not depend on which player's turn it is.

Another important impartial game is **Nim**: given some piles of tokens, choose one pile and remove any positive number of tokens from it; whoever takes the last token wins.



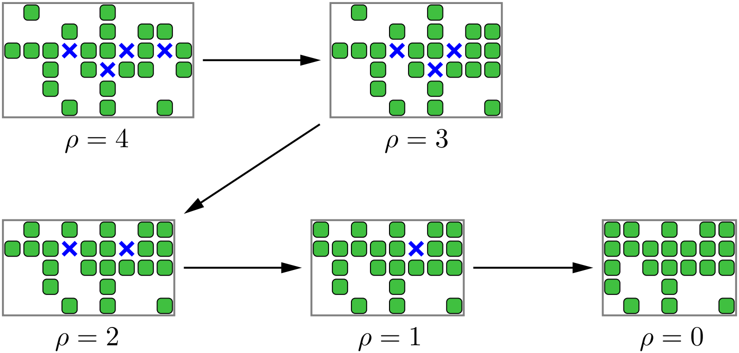
Theorem (Sprague–Grundy, 1935)

Any impartial game is equivalent to a single-pile game of Nim.

Thus, every configuration in our compaction game is equivalent to some Nim configuration, but determining which one is not trivial.

Last-Move-Wins: Potential

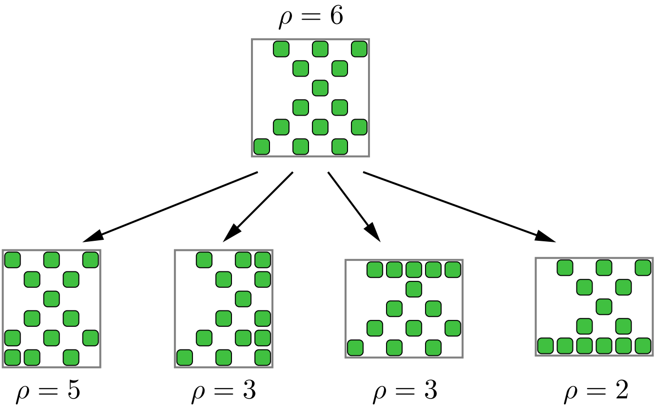
We define the *potential* ρ of a configuration as the minimum number of empty cells in any row plus the minimum number of empty cells in any column (within the bounding box).



The potential decreases at every move; the game ends when $\rho = 0$.

Last-Move-Wins: Potential

Controlling the potential exactly is not trivial. However, trying to make the potential even seems to be a good strategy.

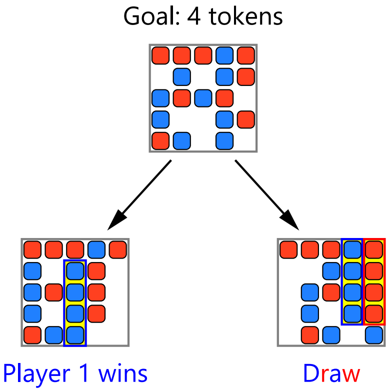


Open problem

Is the compaction game Last-Move-Wins *PSPACE-complete?*

Compaction game: k-in-a-Row

In this compaction game there are tokens of two colors, and each player has to align k tokens of his own color.



Note that the bounding box does not have to shrink at every move. Even the analysis of configurations with $\rho = 0$ is non-trivial.

Compaction game: 2-IN-A-ROW

The version with $k = 2$ is already interesting. Note that, in some configurations, optimal play results in games that loop forever:

Open problem

Can optimal play in 2-IN-A-ROW result in arbitrarily long loops?

A good general strategy for 2-IN-A-ROW seems to be:

- Decrease the distance between tokens of your color located in the same row or column.
- Push tokens of your color between opponent's tokens.
- Disalign the tokens of your opponent's color.

Open problem

Is the compaction game 2-IN-A-ROW PSPACE-complete?

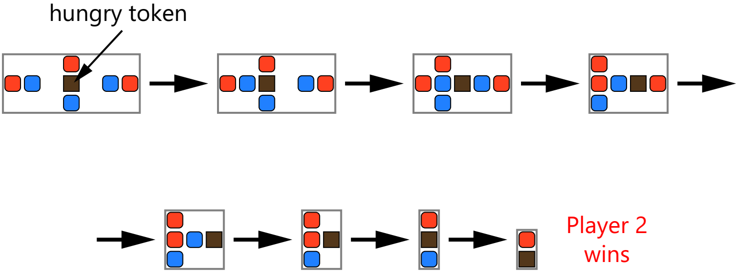
Compaction game: HUNGRY-TOKENS

In this game there are some *hungry tokens* that "eat" other tokens.

Rule 1: Normal tokens push hungry tokens as usual.

Rule 2: Hungry tokens do not push normal tokens, but eat them.

Rule 3: A player loses when all tokens of his color are eaten.



Open problem

Is the compaction game HUNGRY-TOKENS *PSPACE-complete?*

Compaction games: More variations

Some other variations of compaction games to consider:

- In LAST-MOVE-WINS, what if a move pushes tokens from both sides (left+right or up+down)?
- In k-IN-A-ROW, what if Player 1 can only push horizontally and Player 2 only vertically?
- In k-IN-A-ROW, what if Player 1 has to align k of his tokens horizontally and Player 2 vertically?
- In HUNGRY-TOKENS, what if hungry tokens cannot be pushed by nomal tokens (only by the bounding box), but eat any token that is pushed onto them?
- What if all tokens are hungry and can eat each other? Player 1 has to eat all of Player 2's hungry tokens, and vice versa.
- What about compaction games with more than 2 players?