Cyclic Shift Problems on Graphs

WALCOM 2021

Kwon Kham Sai, Ryuhei Uehara, and Giovanni Viglietta

Japan Advanced Institute of Science and Technology (JAIST)

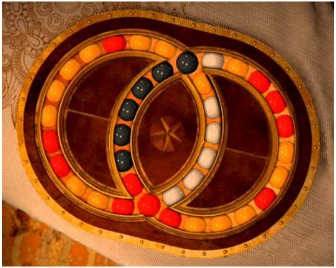
March 2, 2021

Cyclic shift puzzles

This research is inspired by a type of mechanical reconfiguration puzzles based on **cyclic shift** operations:

Cyclic shift puzzles

Each puzzle consists of a set of **colored tokens** that have to be rearranged to form a **target configuration**. The set of allowed moves consists of **token shifts** along predefined cycles.



Our goal is twofold:

- Given a cyclic shift puzzle and an initial configuration, determine which target configurations can be reached.
- Design practical algorithms to solve cyclic shift puzzles in a small number of moves.

Token Shift Problem

If a puzzle has n tokens of c possible colors, a *configuration* is a function mapping "positions" to colors $f\colon \{1,\ldots,n\} \to \{1,\ldots,c\}$.

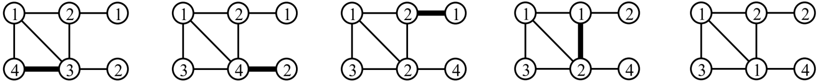
A *cyclic shift puzzle* is a set of cycles γ_1, γ_2, ..., γ_k on $\{1,\ldots,n\}$.

A *shift* along the cycle γ_i (or its inverse γ_i^{-1}) transforms the configuration f into the configuration $\gamma_i \circ f$ (or $\gamma_i^{-1} \circ f$).

The **Token Shift Problem** is to determine a (short) sequence of shifts to transform an initial configuration f_0 into a target one f_t.

Previous work by Yamanaka et al. studied the problem of *swapping* tokens along edges of graphs, as an extension of classical sorting theory.

This problem can be viewed as a special case of the Token Shift Problem where all cycles have length 2, and are given by the edges of a graph.



Tokens of distinct colors

Let us first assume that all n tokens have distinct colors.

Thus, the possible configurations of the puzzle are all the $n!$ **permutations** on n objects: $\pi\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$, which together constitute the **symmetric group** S_n.

In this setting, the cycles γ_1, γ_2, ..., γ_k (and their inverses) generate the subgroup $H = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle \leq S_n$.

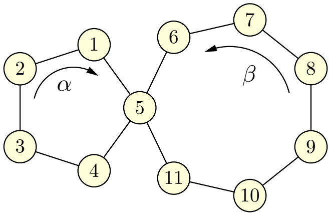
In other words, H is the set of configurations that can be obtained via sequences of shifts starting from the "identity" configuration e.

$\bullet\, e$ H	$\bullet\, \pi_1$ $H\pi_1$	$\bullet\, \pi_2$ $H\pi_2$	$\bullet\, \pi_3$ $H\pi_3$
coset	coset	coset	coset

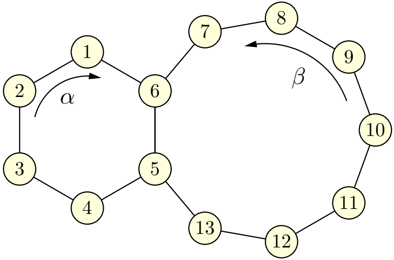
By Lagrange's theorem, the configuration space consists of several disjoint copies of H: its cosets. Each is the set of all configurations that can be reached from some initial configuration π_i.

1-connected and 2-connected puzzles

Let us first focus on puzzles of two basic types:



1-connected



2-connected

For instance, the puzzle on the right has these **generators**:

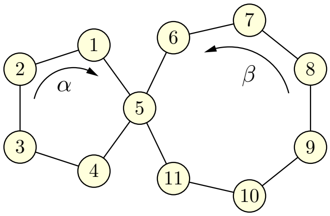
- $\alpha = (1\ 2\ 3\ 4\ 5\ 6)$,
- $\alpha^{-1} = (6\ 5\ 4\ 3\ 2\ 1)$,
- $\beta = (5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13)$,
- $\beta^{-1} = (13\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 5)$.

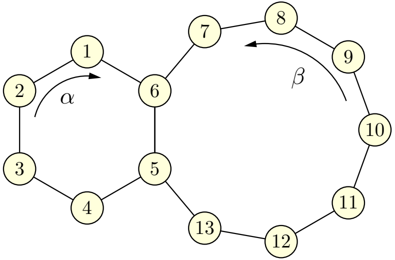
We seek to determine the group $\langle \alpha, \beta \rangle$ and estimate its diameter.

Parity of cycles

Note: a cycle of length ℓ is the composition of $\ell - 1$ transpositions.

Example: $(1\ 2\ 3\ 4\ 5) = (1\ 2)(2\ 3)(3\ 4)(4\ 5)$.





So, if the two cycles α and β both have odd length, they can only generate a subgroup of the **alternating group** A_n (i.e., the group of *even permutations*).

Conversely, if at least one between α and β has even length, then $\langle \alpha, \beta \rangle$ cannot be a subgroup of A_n.

Generators of S_n and A_n

The following facts are folklore, and can be proved by mimicking the *Bubble Sort* algorithm:

(1) $\langle\, (1\ 2\ \ldots\ n),\ (1\ 2)\,\rangle = S_n$.

(2) $\langle\, (1\ 2\ \ldots\ n),\ (1\ 2\ 3)\,\rangle \geq A_n$.*

Any permutation in the group can be generated in $O(n^2)$ steps.

$\Longrightarrow$ If our α and β generate the cycles in either (1) or (2), we can conclude that they generate all of S_n or A_n.

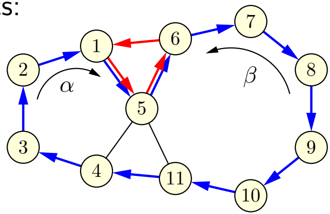
*Obviously, the generated group is S_n if n is even, and A_n if n is odd.

Solving 1-connected puzzles

Theorem

In a 1-connected puzzle, α and β generate A_n if they both have odd length, and S_n otherwise.
Any permutation in the group can be generated in $O(n^2)$ steps.

Proof. $\beta^{-1}\alpha$ is an n-cycle and $\alpha\beta\alpha^{-1}\beta^{-1}$ is a 3-cycle of consecutive elements:



So, $\langle \alpha, \beta \rangle \geq A_n$. If both α and β are even permutations, they cannot generate an odd permutation, and thus $\langle \alpha, \beta \rangle = A_n$.

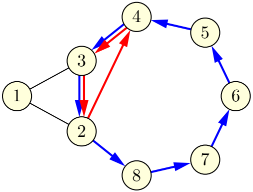
Say α is odd. We can obtain any odd permutation π by generating the even permutation $\pi\alpha$ (as before), and then doing α^{-1}. $\square$

Solving 2-connected puzzles

Theorem

In a 2-connected puzzle with $\alpha = (1\ 2\ 3)$, the generated group is A_n if β has odd length, and S_n if β has even length.
Any permutation in the group can be generated in $O(n^2)$ steps.

Proof. Conjugating α^{-1} by $\alpha^{-1}\beta$, we obtain the 3-cycle $\alpha^{-1}\beta\,\alpha^{-1}\,\beta^{-1}\alpha = (2\ 3\ 4)$ of consecutive elements of β:



So, we can generate any even permutation of $\{2, 3, \ldots, n\}$.

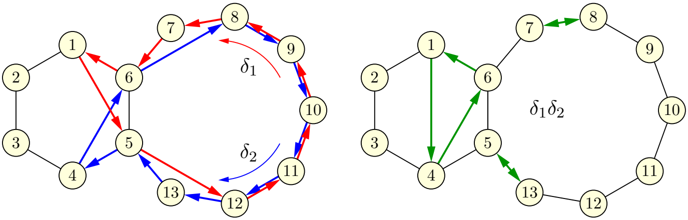
To obtain a given permutation π, first move the correct token $\pi(1)$ in position 1 (possibly shuffling the rest), and then operate on $\{2, 3, \ldots, n\}$ as before (paying attention to parity... details omitted). □

Solving 2-connected puzzles

Theorem

In a 2-connected puzzle, α and β generate A_n if they both have odd length, and S_n otherwise (unless they both have length 4, see later). Any permutation in the group can be generated in $O(n^2)$ steps.

Proof. Conjugating β by $\beta^{-1}\alpha$ and β^{-1} by $\beta\alpha^{-1}$, we obtain two cycles δ_1 and δ_2 of the same length, going in opposite directions:

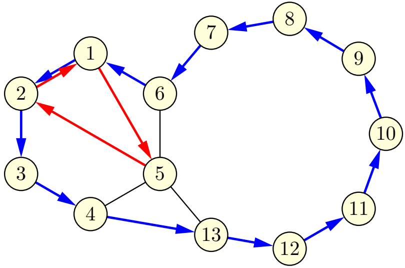


Their composition $\delta_1\delta_2$ is a 3-cycle plus two transpositions.

So, $(\delta_1\delta_2)^2$ is the 3-cycle $(1\ a-2\ a)$, where a is the length of α.

Proof (continued).

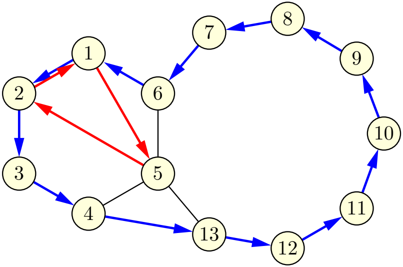
Conjugating $(1\ a-2\ a)$ by α, we obtain the 3-cycle $(1\ 2\ a-1)$.



Note that $(1\ 2\ a-1)$ and $\alpha^{-1}\beta$ form a 2-connected puzzle with a 3-cycle, hence we can apply the previous theorem. □

Proof (continued).

Conjugating $(1\ a-2\ a)$ by α, we obtain the 3-cycle $(1\ 2\ a-1)$.



Note that $(1\ 2\ a-1)$ and $\alpha^{-1}\beta$ form a 2-connected puzzle with a 3-cycle, hence we can apply the previous theorem. □

Only the 2-connected puzzle where α and β have length 4 is left.

As it turns out, in this puzzle we cannot form any 2-cycle or 3-cycle; so, we need a radically new idea...

Outer automorphisms of S_6

It is well known that conjugation by an element of a group G is an **automorphism**. The automorphisms induced by conjugations are called *inner*, and form a subgroup $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.

If $n \neq 6$, the only automorphisms of S_n are the inner ones.

S_6 is an exception: the index of $\mathrm{Inn}(S_6)$ in $\mathrm{Aut}(S_6)$ is 2; so, there are $6! = 720$ *inner* and 720 *outer* automorphisms (Hölder, 1895).

Here is an example of an **outer automorphism** $\psi\colon S_6 \to S_6$ (defined on a generating set for S_6):

$$\psi((1\ 2)) = (1\ 2)(3\ 5)(4\ 6),$$

$$\psi((2\ 3)) = (1\ 6)(2\ 5)(3\ 4),$$

$$\psi((3\ 4)) = (1\ 2)(3\ 6)(4\ 5),$$

$$\psi((4\ 5)) = (1\ 6)(2\ 4)(3\ 5),$$

$$\psi((5\ 6)) = (1\ 2)(3\ 4)(5\ 6).$$

Solving the last 2-connected puzzle

Theorem

In the 2-connected puzzle where α and β have length 4 (so, $n = 6$), the generated group is isomorphic to S_5 (hence it has 6 cosets).

Proof. Idea: transform $\langle \alpha, \beta \rangle$ by ψ and see what group we obtain.

Since ψ is an *isomorphism*, $\langle \alpha, \beta \rangle \cong \langle \psi(\alpha), \psi(\beta) \rangle$.

$\alpha = (1\ 2\ 3\ 4) = (1\ 2)(2\ 3)(3\ 4)$ and

$\beta = (3\ 4\ 5\ 6) = (3\ 4)(4\ 5)(5\ 6)$, thus we have:

$\psi(\alpha) = \psi((1\ 2))\,\psi((2\ 3))\,\psi((3\ 4)) = (1\ 3\ 2\ 4)$,

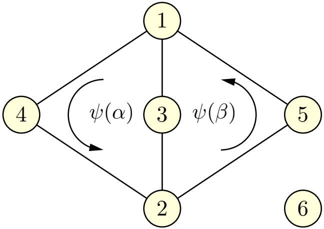
$\psi(\beta) = \psi((3\ 4))\,\psi((4\ 5))\,\psi((5\ 6)) = (1\ 5\ 2\ 3)$.

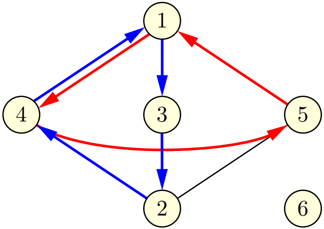
Note: the new generators $\psi(\alpha)$ and $\psi(\beta)$ both leave the token 6 in place, and so they cannot generate a subgroup larger than S_5.

Solving the last 2-connected puzzle

Proof (continued).

The 3-cycle $\psi(\alpha)\,\psi(\beta) = (1\ 5\ 4)$ and the 4-cycle $\psi(\alpha)^{-1}$ form a 2-connected puzzle on $\{1, 2, 3, 4, 5\}$:





By the previous theorem, we know that they generate exactly S_5.

Thus, $\langle \alpha, \beta \rangle$ is an *isomorphic copy* of S_5. A permutation $\pi \in S_6$ is in $\langle \alpha, \beta \rangle$ if and only if $\psi(\pi)$ leaves the token 6 in place. □

Generalized cyclic shift puzzles

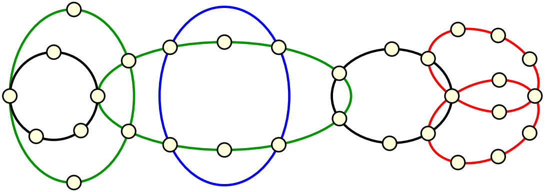
We have characterized the solvable *1-connected* and *2-connected* puzzles, and we have given solutions in $O(n^2)$ moves.

Let us extend our analysis to puzzles with more than two cycles.

Let $\mathcal{C}$ be a set of cycles, and let $\hat{G} = (\mathcal{C}, \mathcal{E})$ be the graph where $\{\gamma_1, \gamma_2\} \in \mathcal{E}$ iff γ_1 and γ_2 induce a 1- or a 2-connected puzzle.

$\mathcal{C}$ forms a **proper cyclic shift puzzle** if there is $\mathcal{C}' \subseteq \mathcal{C}$ such that:

- $\mathcal{C}'$ contains at least two cycles.
- The induced subgraph $\hat{G}[\mathcal{C}']$ is connected.
- Each token is contained in at least one cycle in $\mathcal{C}'$.



Theorem

The configuration group of a proper cyclic shift puzzle with more than 6 tokens is A_n if all cycles have odd length, and S_n otherwise. Any permutation in the group can be generated in $O(n^5)$ steps.

Proof (sketch).

Construct a ("well-behaved") walk W that visits all token locations (possibly more than once) and traverses only edges from cycles in $\mathcal{C}'$.

By our *previous theorems*, any cycle involving 3 consecutive tokens in W can be generated in $O(n^2)$ shifts along cycles in $\mathcal{C}'$.

Carefully composing these 3-cycles $O(n^3)$ times allows us to generate any even permutation (some exceptions are handled separately).

To generate an odd permutation π, shift along an even cycle γ_i, and then generate the even permutation $\gamma_i^{-1}\pi$ as above. □

Cyclic shift puzzles with 2-colored tokens

Let us now consider cyclic shift puzzles where there may be multiple tokens of the **same color**. We will prove the following:

Theorem

Solving the Token Shift Problem in the smallest number of moves is **NP-hard***, even if there are tokens of only 2 colors.*

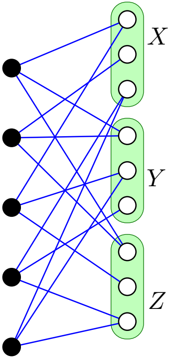
The proof is by a reduction from *3-Dimensional Matching*, which is a classical NP-complete problem.

3-Dimensional Matching (3DM)

Input: Sets X, Y, Z, each of size m. A set of n triplets $T \subseteq X \times Y \times Z$.

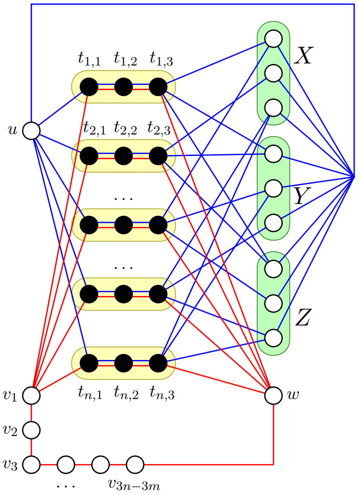
Output: YES iff there is a subset $M \subseteq T$ of size m that covers X, Y, Z.

Reduction from 3DM



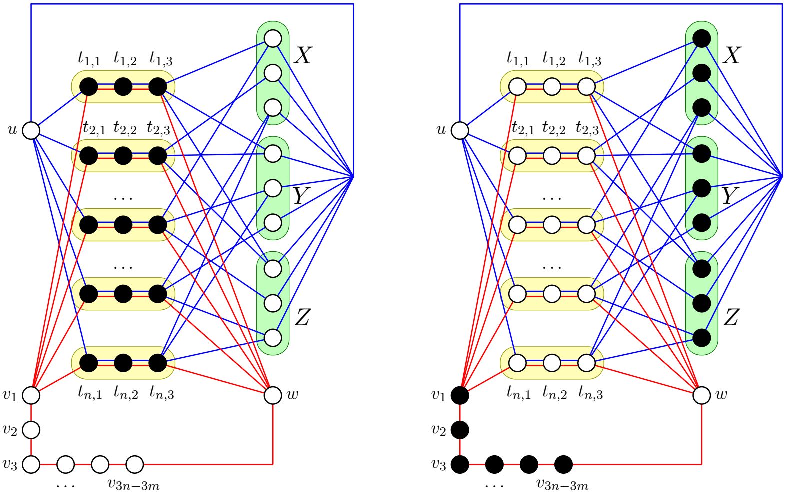
Start from the bipartite graph representing a 3DM instance.

Reduction from 3DM



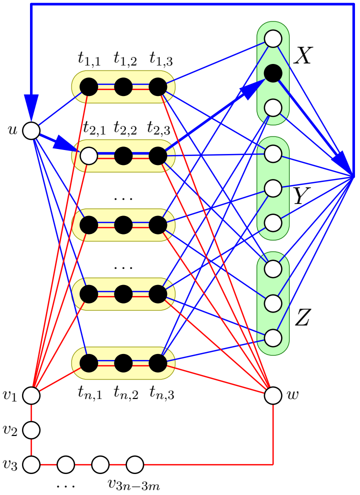
Extend it to a cyclic shift puzzle with black and white tokens.

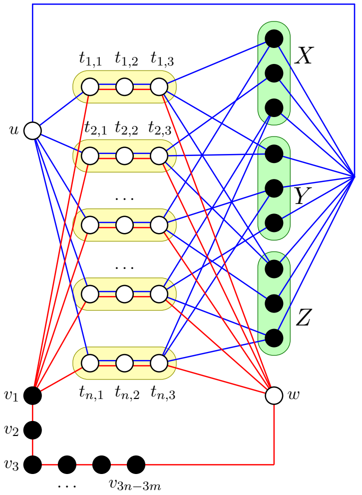
Reduction from 3DM



Our instance of the Token Shift Problem asks to transform the left configuration into the right one in at most $3n$ moves.

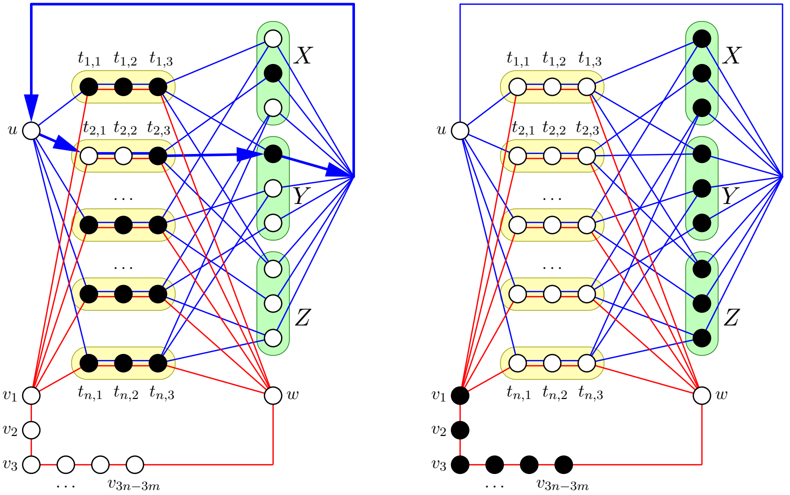
Reduction from 3DM





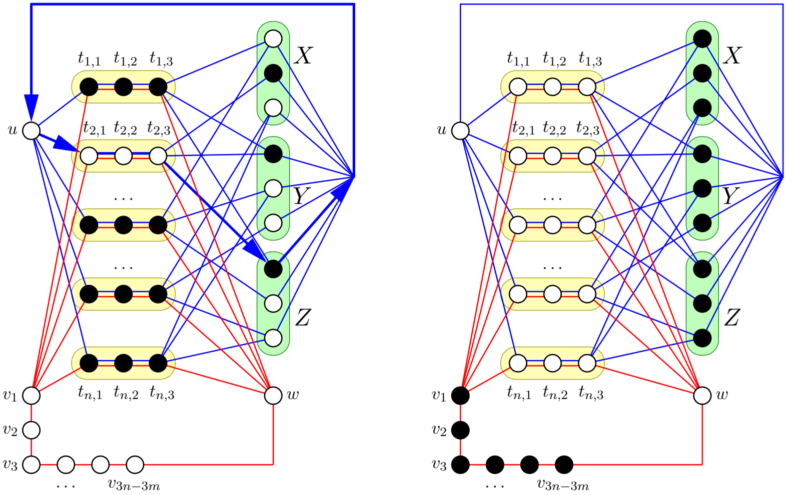
If the 3DM problem has a YES answer, we can pick the triplets in M and shift their tokens along blue cycles to cover X, Y, and Z.

Reduction from 3DM



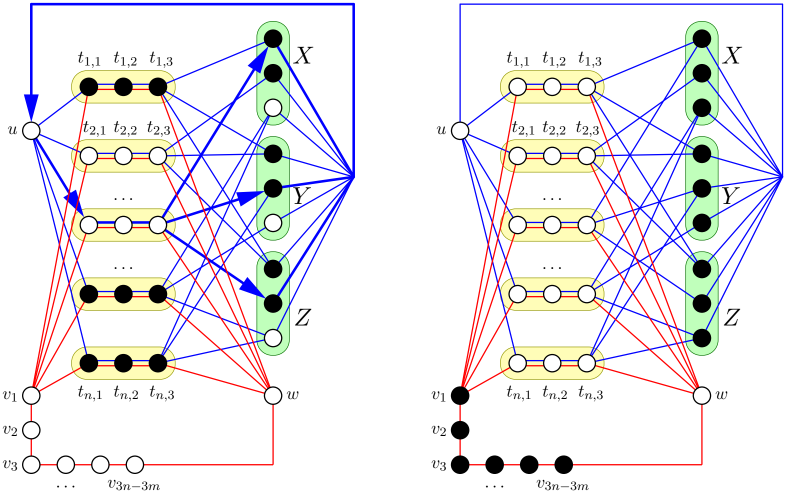
If the 3DM problem has a YES answer, we can pick the triplets in M and shift their tokens along blue cycles to cover X, Y, and Z.

Reduction from 3DM



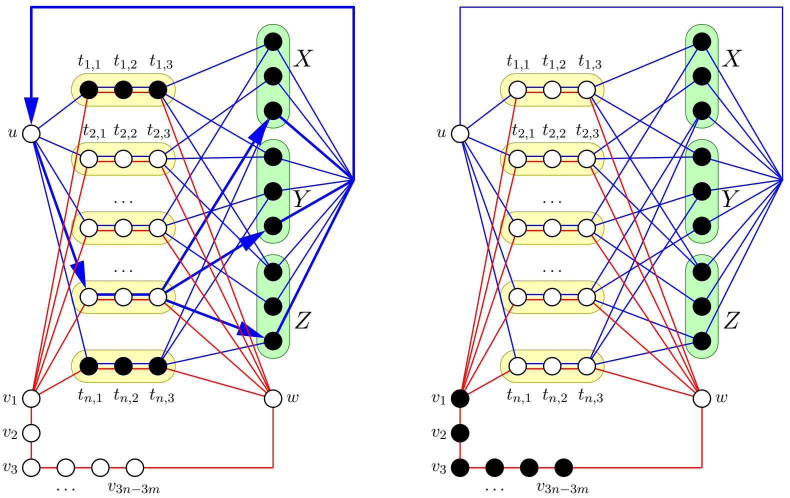
If the 3DM problem has a YES answer, we can pick the triplets in M and shift their tokens along blue cycles to cover X, Y, and Z.

Reduction from 3DM



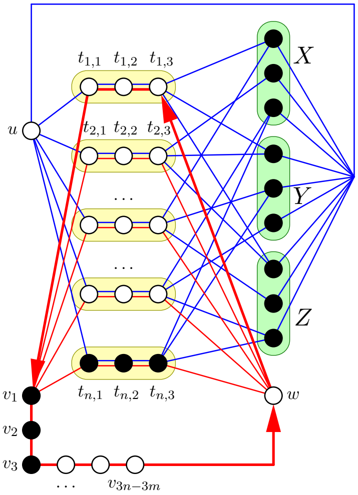
If the 3DM problem has a YES answer, we can pick the triplets in M and shift their tokens along blue cycles to cover X, Y, and Z.

Reduction from 3DM



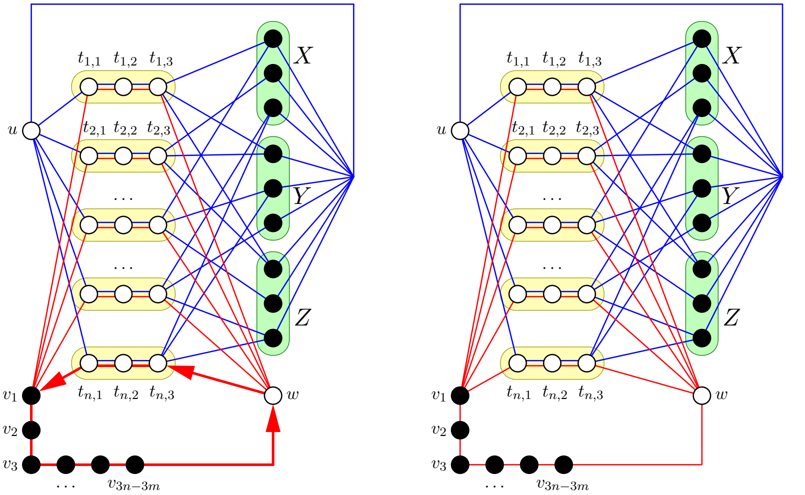
If the 3DM problem has a YES answer, we can pick the triplets in M and shift their tokens along blue cycles to cover X, Y, and Z.

Reduction from 3DM



Then, we shift the tokens from the other triplets along red cycles.
The total number of moves is $3n$.

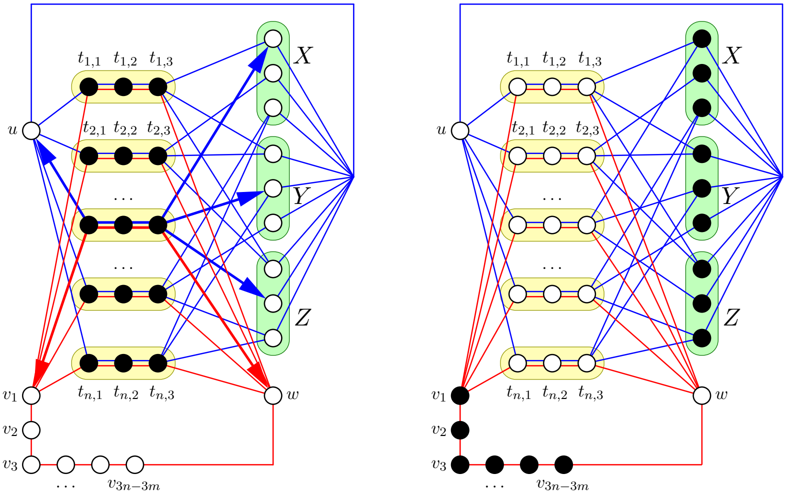
Reduction from 3DM



Then, we shift the tokens from the other triplets along red cycles.

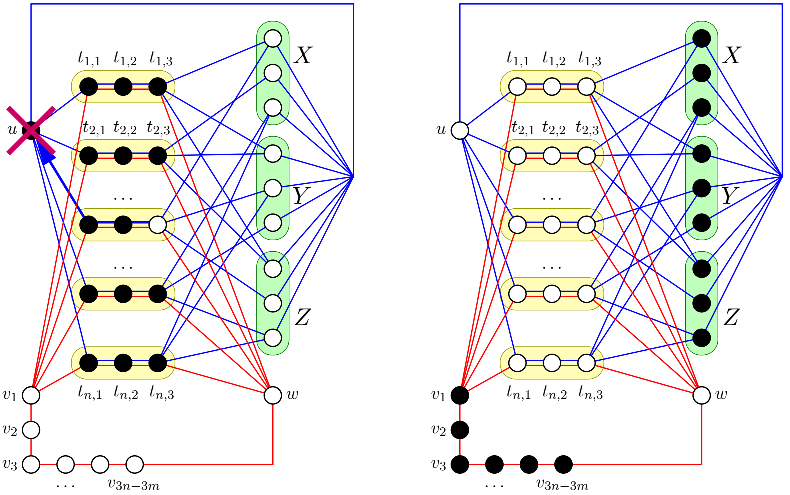
The total number of moves is $3n$.

Reduction from 3DM



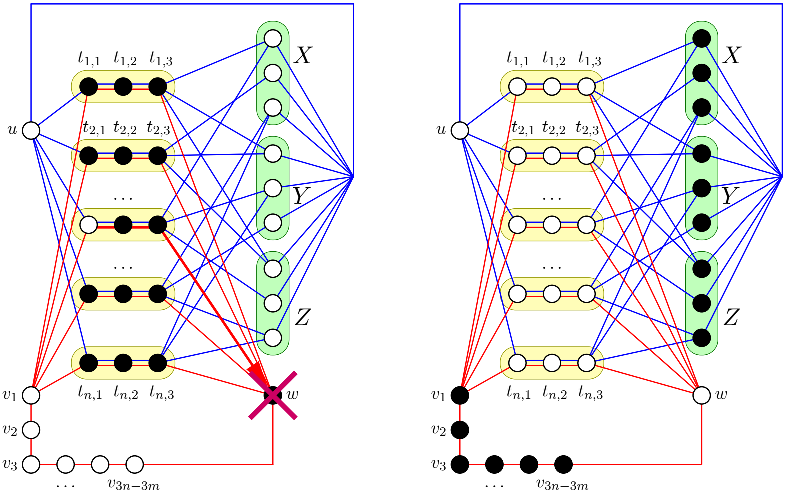
Conversely, assume that the tokens can be rearranged in $3n$ moves. Then, each move must get one black token out of the yellow area.

Reduction from 3DM



It follows that no black token can ever be found in u.
Otherwise, u will always contain a black token afterward.

Reduction from 3DM



For the same reason, no black token can ever be found in w. Now it is easy to prove that the 3DM instance must have a YES answer.

Summary and open problems

We have studied the **Token Shift Problem** in two settings:

- For a large class of puzzles with tokens of distinct colors, we characterized the reachable configurations, and we gave an algorithm to reach them in a polynomial number of shifts.
- If multiple tokens are allowed to have the same color, we showed that minimizing the number of shifts is NP-hard.

Some **open problems** are left for future research:

- Improve the $O(n^5)$ upper bound of our algorithm.
- Extend the analysis to cyclic shift puzzles that are not proper.
- Is the minimization problem PSPACE-complete?
- Is it NP-hard for planar graphs, or for graphs of small maximum degree?