Edge-guarding Orthogonal Polyhedra

(23^{rd} Canadian Conference on Computational Geometry)

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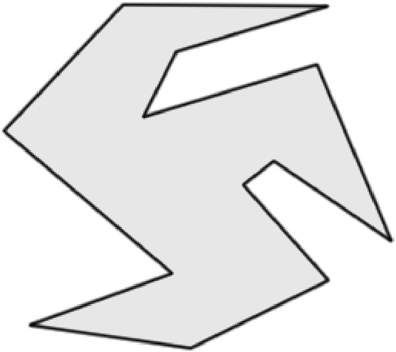
Toronto - August 12^{th}, 2011

Art Gallery Problem

- **Planar version:** Given a polygon, choose a minimum number of vertices that collectively see its whole interior.



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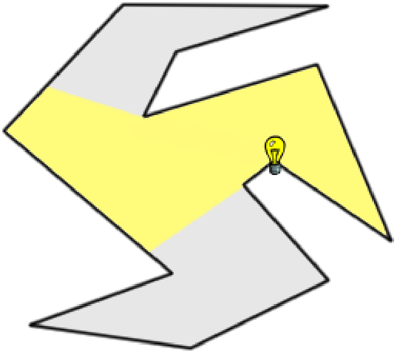

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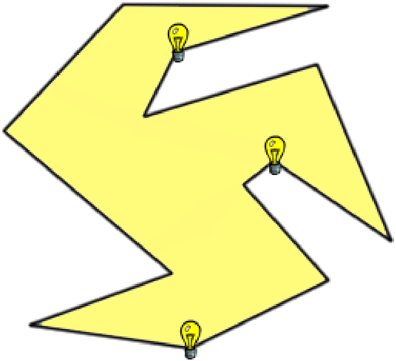

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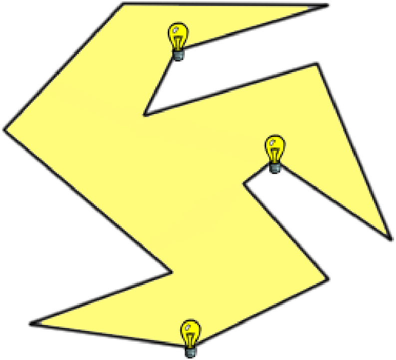
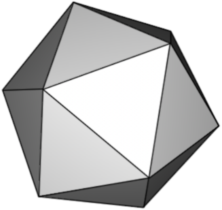


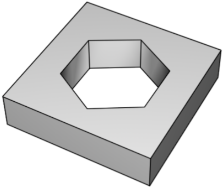
- **Problem:** Generalize to *orthogonal polyhedra*.

Terminology

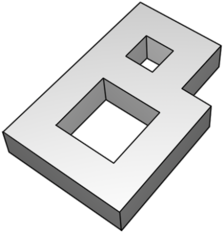
Polyhedra



genus 0

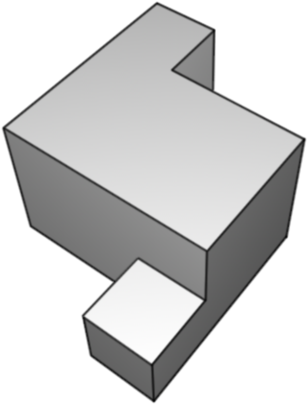


genus 1

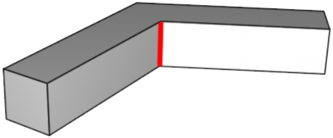


genus 2

Terminology



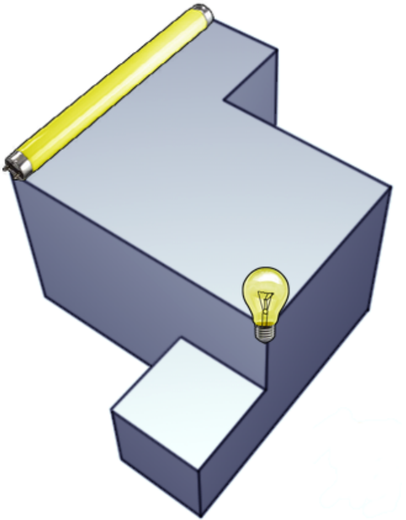
Orthogonal polyhedron



Reflex edge

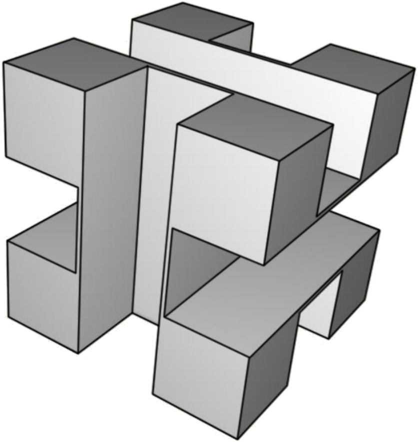
Guarding polyhedra

- Vertex guards vs. edge guards.



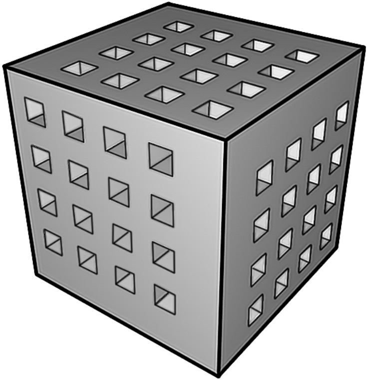
Vertex-guarding orthogonal polyhedra

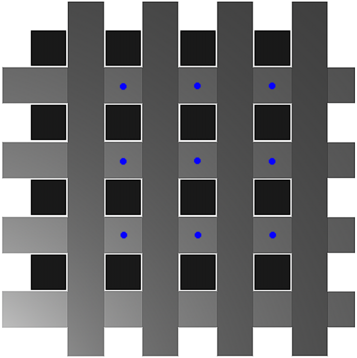
- The Art Gallery Problem for vertex guards is unsolvable on some orthogonal polyhedra.



- Some points in the central region are invisible to all vertices.

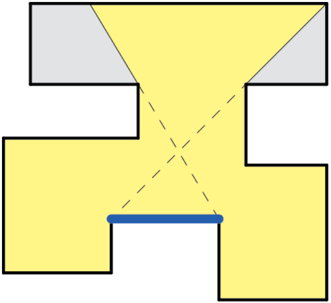
- Some orthogonal polyhedra require $\Omega(n^{3/2})$ point guards.

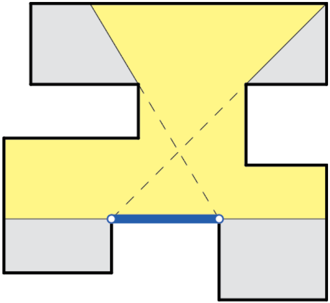




Edge guards

- Closed edge guards vs. open edge guards.





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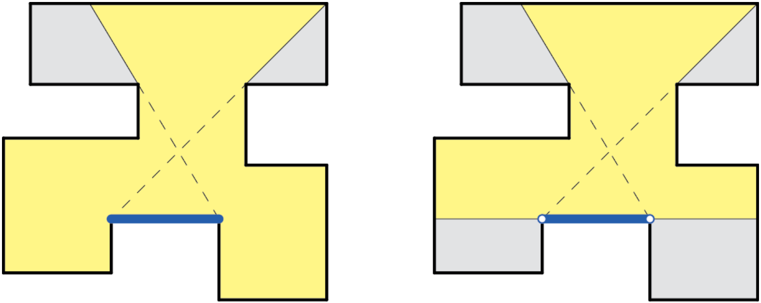

- **Motivation for open edge guards:** Each illuminated point receives light from a non-degenerate subsegment of a guard.

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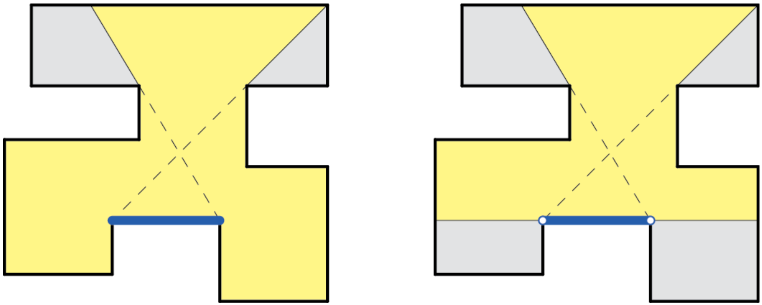
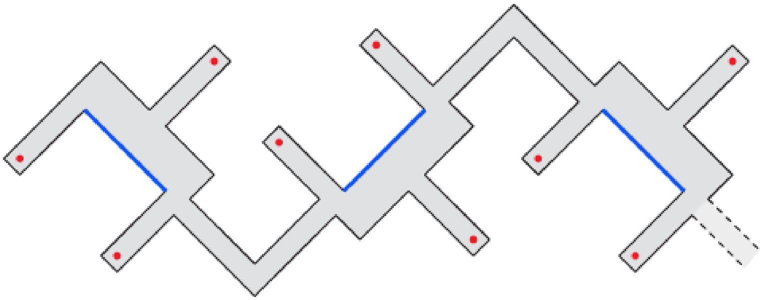


- **Motivation for open edge guards:** Each illuminated point receives light from a non-degenerate subsegment of a guard.
- **Problem:** How much more powerful are closed edge guards?

Closed vs. open edge guards

- Closed edge guards are at least 3 times more powerful.
  - No open edge can see more than one red dot.



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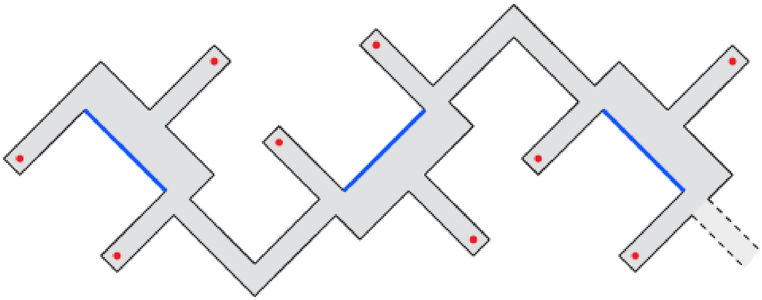
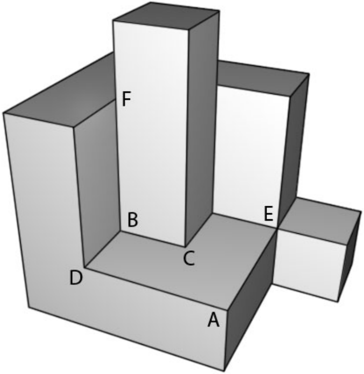


- Is this lower bound tight?

Closed vs. open edge guards

- Each endpoint of a closed edge guard can be replaced by an adjacent open edge.
  - Case analysis on all vertex types.



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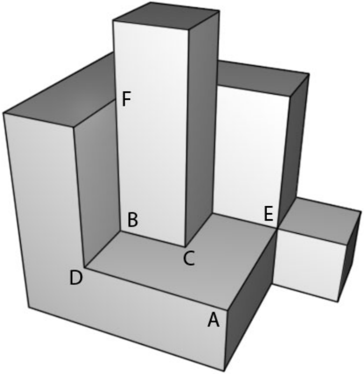

- Hence each closed edge guard can be replaced by 3 open edge guards, and our previous bound is tight.

Bounding edge guards

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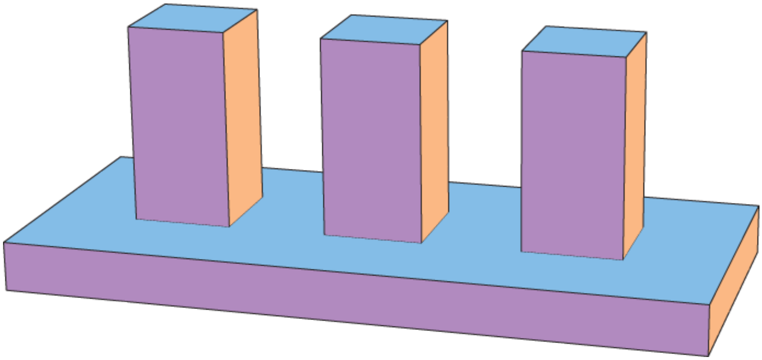
- Typically, we content ourselves with upper bounds on the minimum number of guards.
- Our parameters for bounding edge guards in orthogonal polyhedra are the total number of edges **e** and the number of reflex edges **r**.

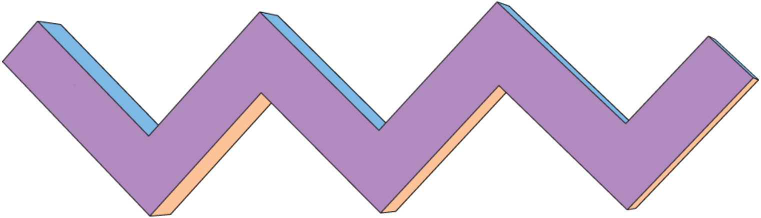
Lower bound

- Asymptotically, $\frac{e}{12}$ edge guards may be necessary.



Lower bound

- Asymptotically, $\frac{\mathbf{r}}{\mathbf{2}}$ reflex edge guards may be necessary.



Upper bound

- **Observation:** Any polyhedron is guarded by the set of its edges.
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State of the art (Urrutia)

Any orthogonal polyhedron is guardable by $\frac{\mathbf{e}}{\mathbf{6}}$ *closed edge guards.*

- Can it be lowered and extended to open edge guards?

Improving the upper bound

Theorem

Any orthogonal polyhedron is guardable by $\frac{\mathbf{e}+\mathbf{r}}{\mathbf{12}}$ *open edge guards.*

Improving the upper bound

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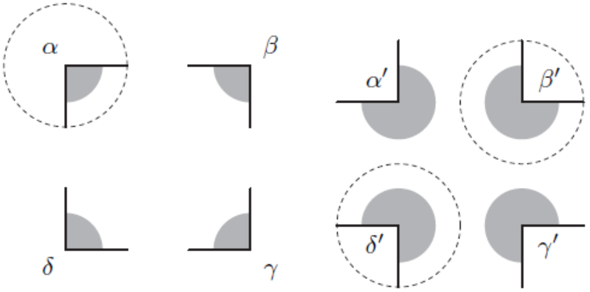
Proof.

We select a coordinate axis X and only place guards on X-parallel edges.

There are 8 types of X-parallel edges, and we place guards on the circled ones (X axis pointing toward the audience):



Improving the upper bound

There are 4 symmetric ways of picking edge types:

$$\alpha + \beta' + \delta',$$
$$\gamma + \beta' + \delta',$$
$$\beta + \alpha' + \gamma',$$
$$\delta + \alpha' + \gamma'.$$

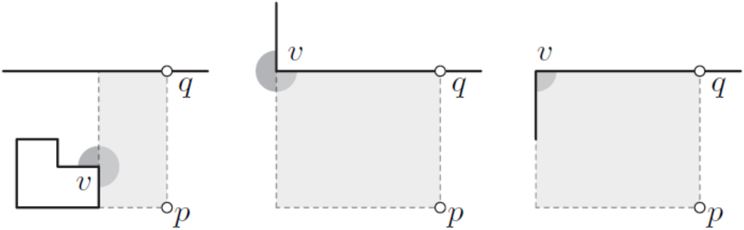
The sum is

$$\alpha + \beta + \gamma + \delta + 2\alpha' + 2\beta' + 2\gamma' + 2\delta' \quad = \quad e_x + r_x.$$

Hence, one of the 4 choices picks at most $\frac{e_x + r_x}{4}$ edges.

By selecting the axis X that minimizes the sum $e_x + r_x$, we place at most $\frac{e+r}{12}$ guards.

Indeed, every X-orthogonal section is guarded:



For a given p, pick the maximal segment pq and slide it to the left, until it hits a vertex v, which corresponds to a selected edge. □

Improving the upper bound

Theorem

For every orthogonal polyhedron of genus g,

$$\frac{1}{6}e + 2g - 2 \leqslant r \leqslant \frac{5}{6}e - 2g - 12$$

holds. Both inequalities are tight for every g.

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Corollary

$\mathbf{\frac{11}{72}e - \frac{g}{6} - 1}$ *open edge guards are sufficient to guard any orthogonal polyhedron.*

Corollary

$\mathbf{\frac{7}{12}r - g + 1}$ *open edge guards are sufficient to guard any orthogonal polyhedron.*

Concluding remarks

- We showed that closed edge guards are **3** times more powerful than open edge guards, for orthogonal polyhedra.
- We lowered the upper bound on the number of edge guards from $\frac{\mathbf{e}}{\mathbf{6}}$ to $\frac{\mathbf{11}}{\mathbf{72}}\mathbf{e}$, whereas the best known lower bound is $\frac{\mathbf{e}}{\mathbf{12}}$.
- We gave the new upper bound $\frac{\mathbf{7}}{\mathbf{12}}\mathbf{r}$, whereas the best known lower bound is $\frac{\mathbf{r}}{\mathbf{2}}$.

Further research

Conjecture

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Any orthogonal polyhedron is guardable by $\frac{\mathbf{e}}{\mathbf{12}}$ *edges and* $\frac{\mathbf{r}}{\mathbf{2}}$ *reflex edges.*

- How to bound the number of guards in terms of r, while actually placing them on reflex edges only?

Theorem (O'Rourke)

Any orthogonal prism is guardable by $\left\lfloor \frac{\mathbf{r}}{\mathbf{2}} \right\rfloor + \mathbf{1}$ *reflex edge guards.*

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Theorem (O'Rourke)

Any orthogonal prism is guardable by $\left\lfloor \frac{\mathbf{r}}{\mathbf{2}} \right\rfloor + \mathbf{1}$ *reflex edge guards.*

Theorem

Any orthogonal polyhedron with reflex edges in just two directions is guardable by $\left\lfloor \frac{\mathbf{r}}{\mathbf{2}} \right\rfloor + \mathbf{1}$ *reflex edge guards.*

Corollary

Any orthogonal polyhedron is guardable by $\left\lceil \frac{\mathbf{2}}{\mathbf{3}}\mathbf{r} \right\rceil$ *reflex edge guards.*

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- What if we consider polyhedra with faces in 4 different directions?
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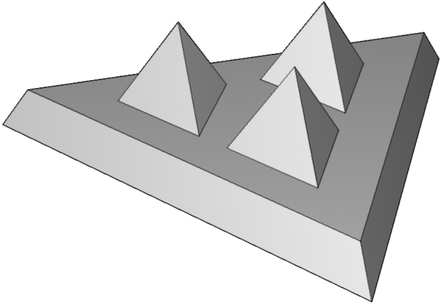

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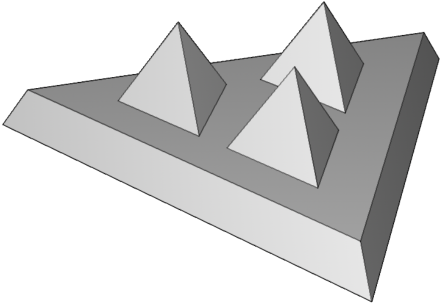

Theorem

Any such polyhedron is guardable by $\frac{\mathbf{e+r}}{\mathbf{6}}$ open edge guards.

Thank you!