Face-Guarding Polyhedra

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The Window Positioning Problem
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**Problem:** What is the minimum number $g$ of faces that have to be selected to guard the interior of a polyhedron with $f$ faces?
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**Theorem (Souvaine–Veroy–Winslow, 2011)**

For (closed) face guards in orthogonal polyhedra,

\[
\left\lfloor \frac{f}{7} \right\rfloor \leq g(f) \leq \left\lfloor \frac{f}{6} \right\rfloor
\]

**Theorem (Souvaine–Veroy–Winslow, 2011)**

For (closed) face guards in general polyhedra,

\[
\left\lfloor \frac{f}{5} \right\rfloor \leq g(f) \leq \left\lfloor \frac{f}{2} \right\rfloor
\]
Open problems (Souvaine–Veroy–Winslow, 2011)

- Study *open*, as well as closed, face guards.
- Study polyhedra of arbitrary genus.
- Determine the complexity of minimizing face guards.
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- Determine the complexity of minimizing face guards.

Theorem (Iwamoto–Kitagaki–Morita, 2012)

*Minimizing face guards in polyhedral terrains in NP-hard.*

(Not applicable to polyhedra)
Guards patrolling on faces

Face guards: a model for guards *roaming* over a face?
An edge guard may not be “replaceable” by finitely many of its points.
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The right endpoint must be a limit point of the guarding set.
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Some faces represent guards patrolling on paths of quadratic “complexity” (or quadratically many segment guards).
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![Diagram of face-guarding polyhedra]

Face-Guarding Polyhedra
Guards patrolling on faces

Some faces represent guards patrolling on paths of quadratic “complexity” (or quadratically many segment guards).

The face guard model is insensitive to the complexity of a guard’s path, and is better suited as a model for entities that are naturally constrained to live on a face (e.g., flat windows).
Theorem

For open or closed face guards and c-oriented polyhedra,

\[ g(f) \leq \left\lfloor \frac{f}{2} - \frac{f}{c} \right\rfloor \]
Upper bound on face guard numbers

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For open or closed face guards and $c$-oriented polyhedra,

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For open or closed face guards and $c$-oriented polyhedra,

$$g(f) \leq \left\lfloor \frac{f}{2} - \frac{f}{c} \right\rfloor$$

- For orthogonal polyhedra ($c = 3$): $\left\lfloor \frac{f}{6} \right\rfloor$ face guards.
- For 4-oriented polyhedra: $\left\lfloor \frac{f}{4} \right\rfloor$ face guards.
- For general polyhedra ($c = f$): $\left\lfloor \frac{f}{2} \right\rfloor - 1$ face guards.
Lower bounds for orthogonal polyhedra

For closed face guards in orthogonal polyhedra, \( g(f) \geq \lfloor f/7 \rfloor \).
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Theorem

For open face guards in orthogonal polyhedra, \( g(f) = \lfloor f/6 \rfloor \).
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Face-Guarding Polyhedra
Theorem

For open face guards in orthogonal polyhedra, \( g(f) = \left\lfloor \frac{f}{6} \right\rfloor \).
Theorem

For closed face guards in 4-oriented polyhedra, $g(f) \geq \lfloor f/5 \rfloor$. 

Face-Guarding Polyhedra
Lower bounds for 4-oriented polyhedra

Theorem

For closed face guards in 4-oriented polyhedra, \( g(f) \geq \lfloor f \rfloor \).
For closed face guards in 4-oriented polyhedra, \( g(f) \geq \left\lfloor \frac{f}{5} \right\rfloor \).
Theorem
For open face guards in 4-oriented polyhedra, $g(f) = \lfloor f/4 \rfloor$. 

Face-Guarding Polyhedra
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Face-Guarding Polyhedra
Theorem

The minimum number of (open or closed) face guards is NP-hard to approximate within a factor of $\Omega(\log f)$.
Hardness of approximation

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Reduction from Set Cover.

Let $\mathcal{U} = \{1, \cdots, n\}$ and $S \subseteq \mathcal{P}(\mathcal{U})$, with $|S| = m$.

We construct an orthogonal polyhedron with $f = O(mn)$ faces that is guardable by $k + 1$ faces iff $\mathcal{U}$ is the union of $k$ elements of $S$.

The theorem follows from the inapproximability of Set Cover.
The minimum number of (open or closed) face guards is NP-hard to approximate within a factor of $\Omega(\log f)$.

$\mathcal{U} = \{1, 2, 3, 4\}$
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$S_1 = \{2, 4\}$
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$S_3 = \{2\}$
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Membership in NP

Traditional approach: partition a polygon into polynomially many regions, each of which is either visible or invisible to any guard.
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Membership in NP

Even in orthogonal polyhedra, the region visible to a face may be bounded by a quadric surface.
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Membership in NP

The boundary of the visible area of a face is a piecewise quadric determined by lines passing through pairs or triplets of edges of the polyhedron.

The quadric surface determined by three edges may be:

- a plane, if two of the edges are parallel;
- a hyperbolic paraboloid, if the edges are parallel to a plane;
- a hyperboloid of one sheet, otherwise.
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- a plane, if two of the edges are parallel;
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There exist algebraic methods to compute intersections of quadric surfaces, but they involve parameterizations containing radicals.
Open problems

- Is the minimization problem in NP?
  - Is it $\Theta(\log f)$-approximable?
  - What about its restriction to orthogonal polyhedra?

- Are $\lfloor f \rfloor$ closed face guards sufficient for every orthogonal polyhedron?
- Are $\lfloor \frac{f}{2} \rfloor$ open face guards sufficient for every polyhedron?
Open problems

- Is the minimization problem in NP?
  Is it $\Theta(\log f)$-approximable?
  What about its restriction to orthogonal polyhedra?

- Are $\left\lfloor \frac{f}{7} \right\rfloor$ closed face guards sufficient for every orthogonal polyhedron?

- Are $\left\lfloor \frac{f}{5} \right\rfloor$ closed face guards sufficient for every polyhedron?

- Are $\left\lfloor \frac{f}{4} \right\rfloor$ open face guards sufficient for every polyhedron?
C. Iwamoto, Y. Kitagaki, and K. Morita.  
*Finding the Minimum Number of Face Guards is NP-Hard.*  

D. L. Souvaine, R. Veroy, and A. Winslow.  
*Face Guards for Art Galleries.*  

G. Viglietta.  
*Guarding and Searching Polyhedra.*  