Computing in Anonymous Dynamic Networks Is Linear

FOCS 2022

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Dynamic networks

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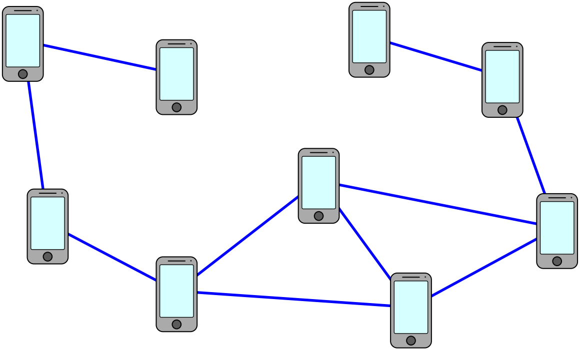

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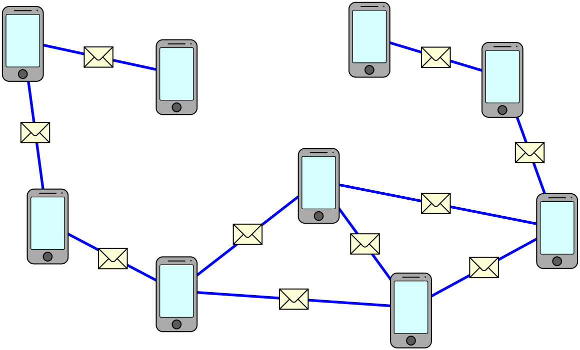

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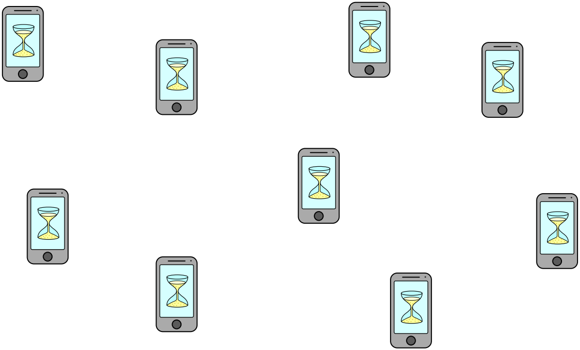

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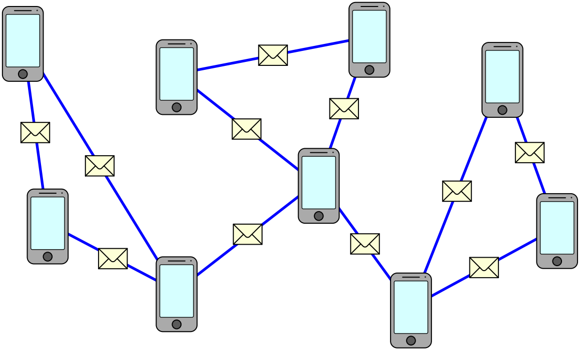

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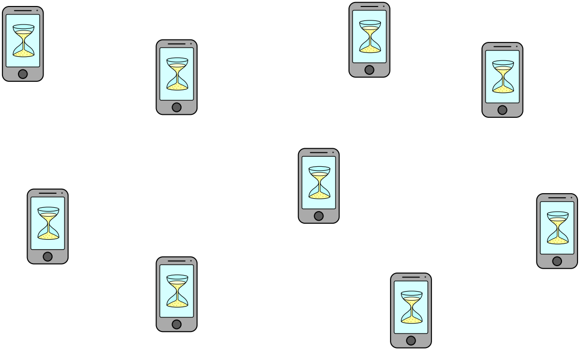

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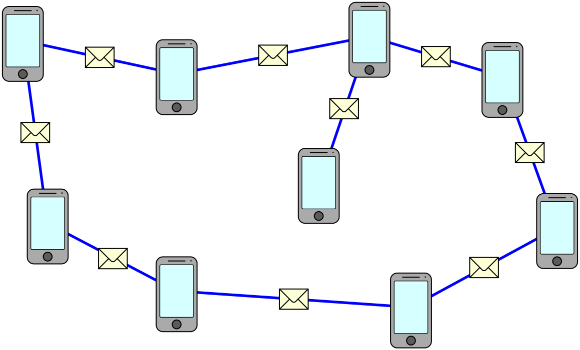

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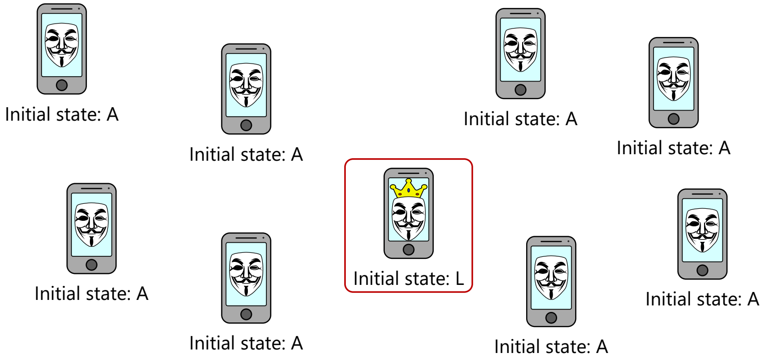
Counting anonymous agents with a Leader

A common assumption is that the dynamic network is *anonymous*, i.e., all agents start in the same state, except one: the *Leader*.



The *complete problem* in this model is the **Counting Problem:** Eventually, all agents must know the total number of agents, n.
(If agents have inputs, also compute how many agents have each input.)

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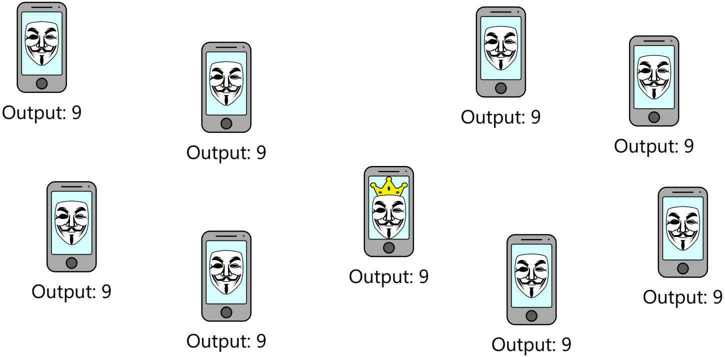

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Literature on the Counting Problem

- **Michail et al.:** Looks impossible! (SSS 2013)
- **Di Luna et al.:** Solvable in $O(e^{N^2} N^3)$ rounds (ICDCN 2014)
- **Di Luna–Baldoni:** $O(n^{n+4})$ rounds (OPODIS 2015)
- **Kowalski–Mosteiro:** $O(n^5 \log^2 n)$ rounds (ICALP 2018 Best Paper)
- **Kowalski–Mosteiro:** $O(n^{4+\epsilon} \log^3 n)$ rounds (ICALP 2019)
- **This work:** $3n$ rounds

Symbols:

- n: number of agents in the network (unknown)
- N: upper bound on n (unknown, except in ICDCN 2014)

Statement of results

Theorem (Main result)

For the Counting Problem, we have:

- *Stabilizing algorithm in* $2n$ *rounds (no termination).*
- *Terminating algorithm in* $3n$ *rounds.*
- *Lower bound of* $2n$ *rounds (for stabilization and termination).*

Local memory, local computation time, and message size are polynomial in n*. Also works if the network is a multi-graph.*

Actually, the theorem holds not only for the Counting Problem, but for *all* problems computable in anonymous (dynamic) networks.

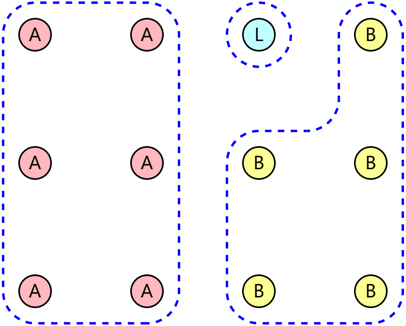
These are precisely the *multi-aggregate* functions f:

- Agent p outputs $f(x_p, \mu)$,
- where x_p is the input of agent p,
- and μ is the multi-set of all inputs.

Computability

General computation

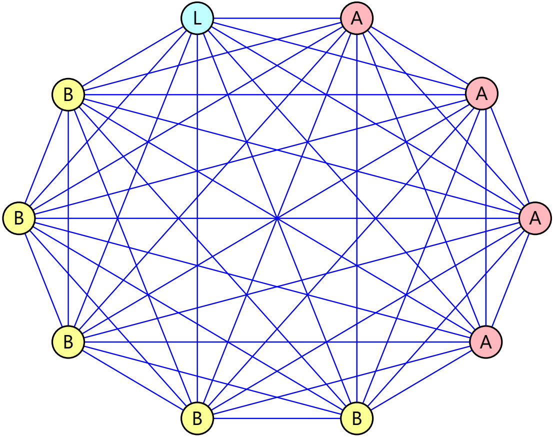
In general, we may assume that each agent has an *input* and has to compute an *output* depending on the entire network's inputs.



Agents with the same input are still indistinguishable (anonymous).

General computation

If the network is the complete graph at every round, all agents with the same input will always have the same internal state.



Thus, an agent's output can only depend on its input and the *number* of agents having each input.

Completeness of the Generalized Counting Problem

Thus, only the *multi-aggregate* functions can be computed.

Observation

If a function is computable in an anonymous dynamic network (with a unique Leader), it must be a multi-aggregate function.

Examples: The average, maximum, minimum, sum, mode, variance, and most statistical functions are (multi-)aggregate.

Generalized Counting Problem: Eventually, all agents must know how many agents have each input.

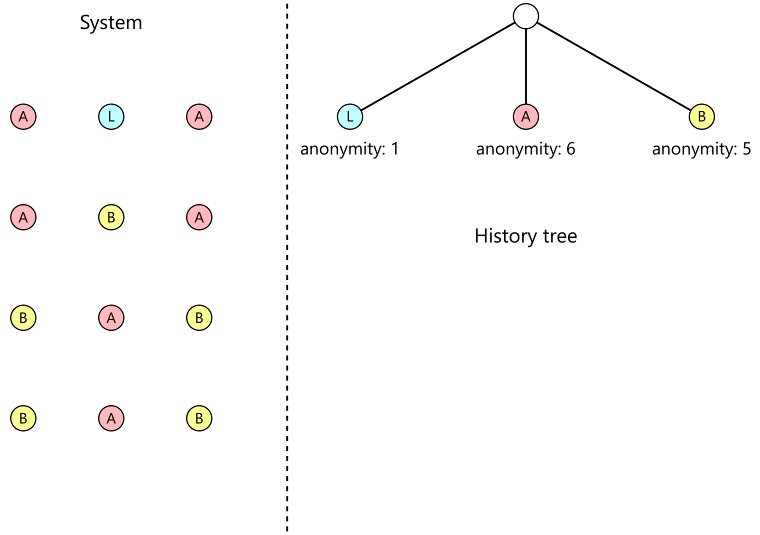
Observation

If the Generalized Counting Problem is solvable in $f(n)$ rounds, then every multi-aggregate function is computable in $f(n)$ rounds.

History Trees

History trees

We introduce the *history tree* as our main tool of investigation.



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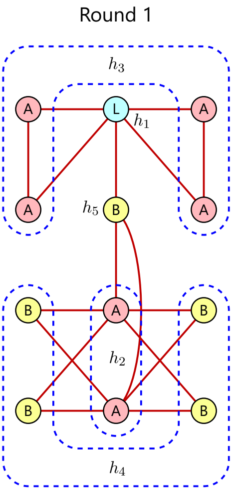

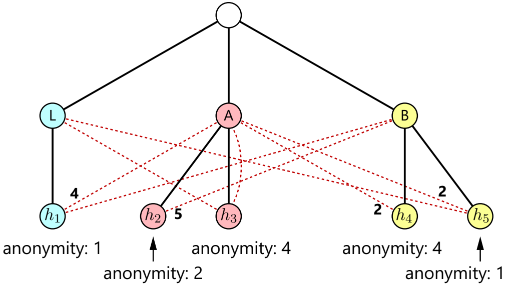

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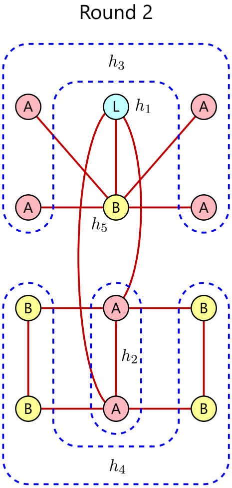

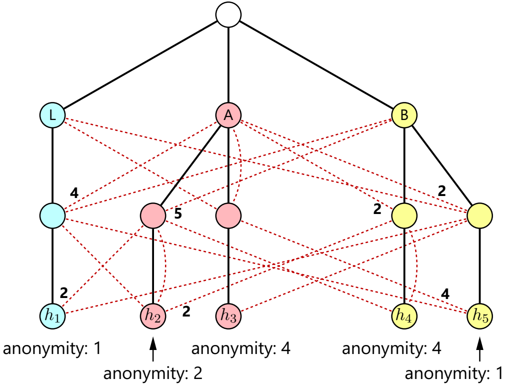

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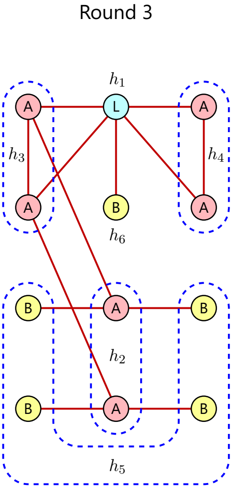

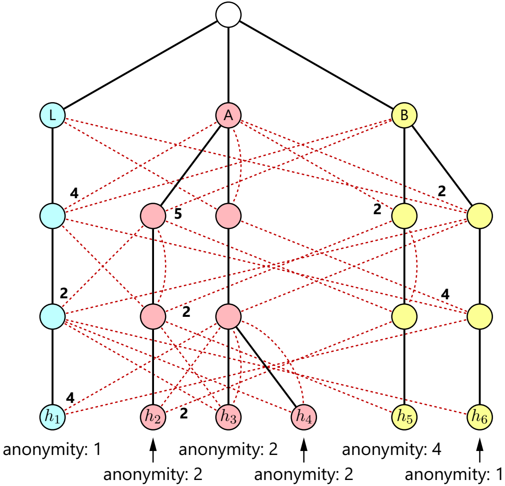

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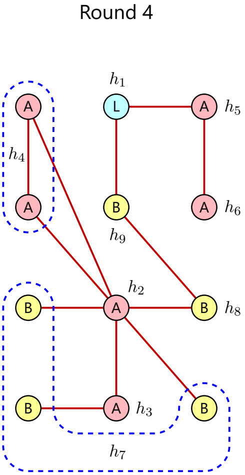
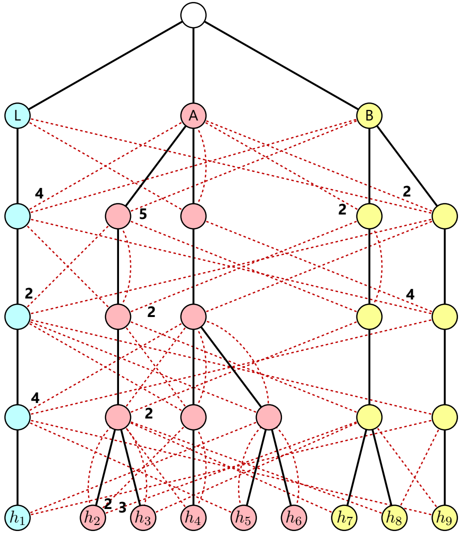
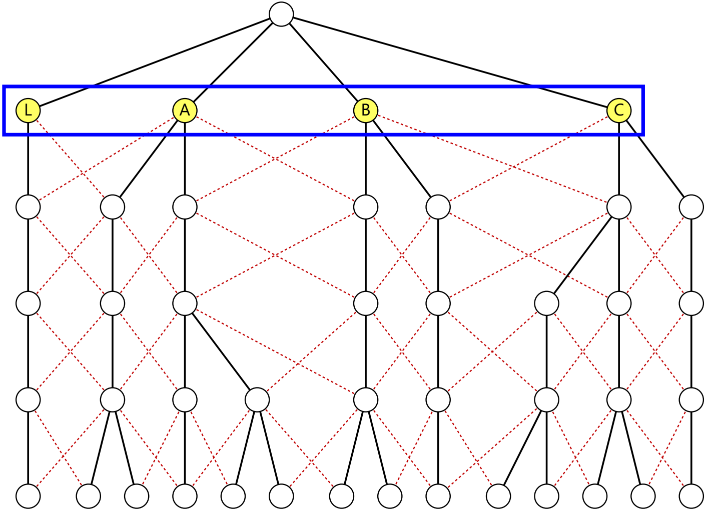




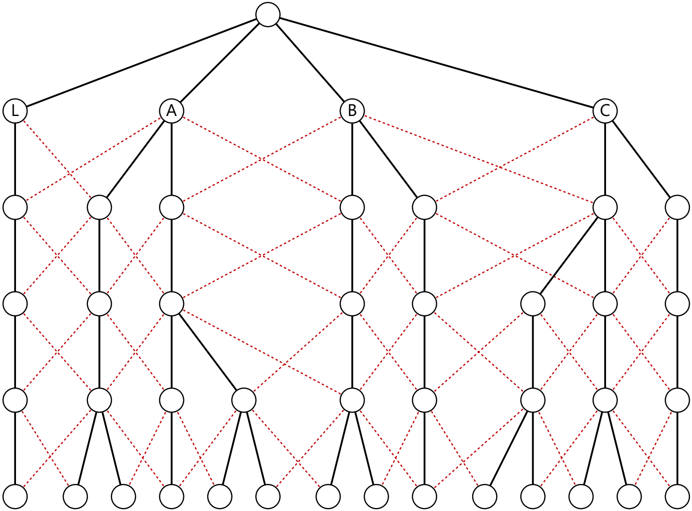
Generalized Counting Problem revisited

Solving the Generalized Counting problem amounts to finding the anonymities of the nodes in the first level of the history tree.



View of a history tree

At any point in time, an agent only has a *view* of the history tree.



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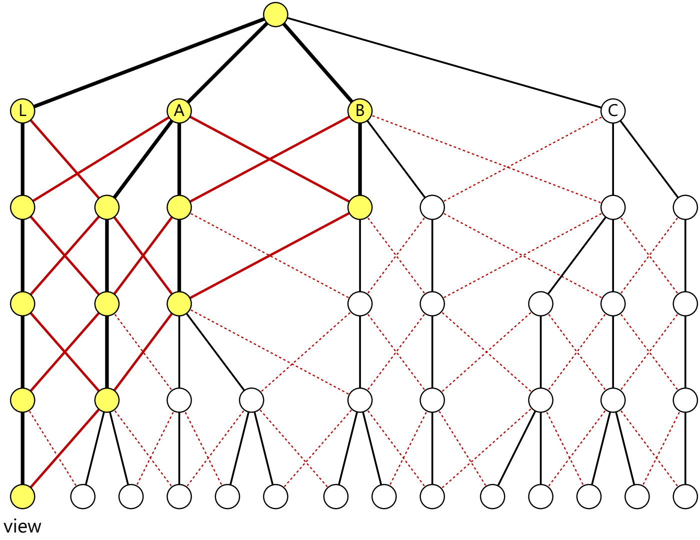

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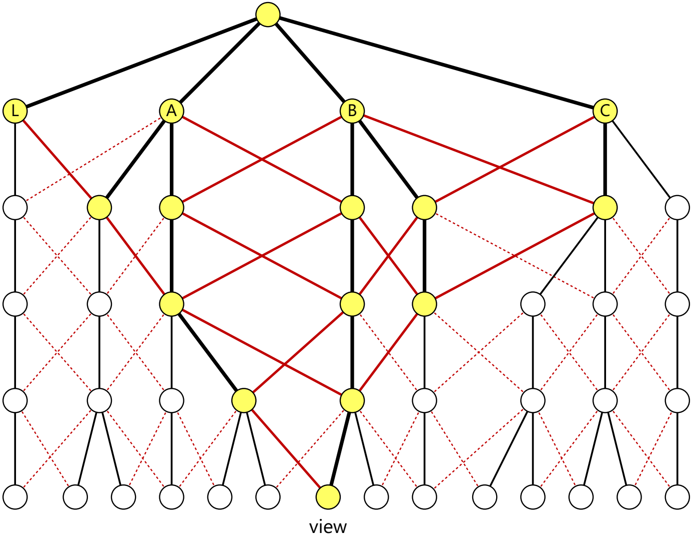


Views as internal states and messages

An agent's view summarizes its whole *history* up to some round.

Observation

Without loss of generality, we may assume that an agent's internal state coincides with its view of the history tree.

Observation

Without loss of generality, we may assume that an agent broadcasts its own internal state at every round.

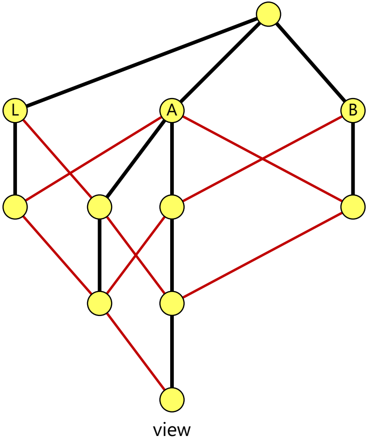
This is good because, at round i, the size of a view is only $O(i^3)$.

Observation

If a problem is solvable in a polynomial number of rounds, it can be solved by using a polynomial amount of local memory and sending messages of polynomial size.

Updating the view

An agent updates its internal state by *merging* its view with the views it receives from its neighbors.



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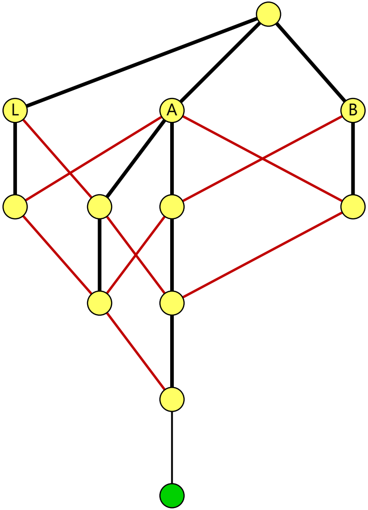

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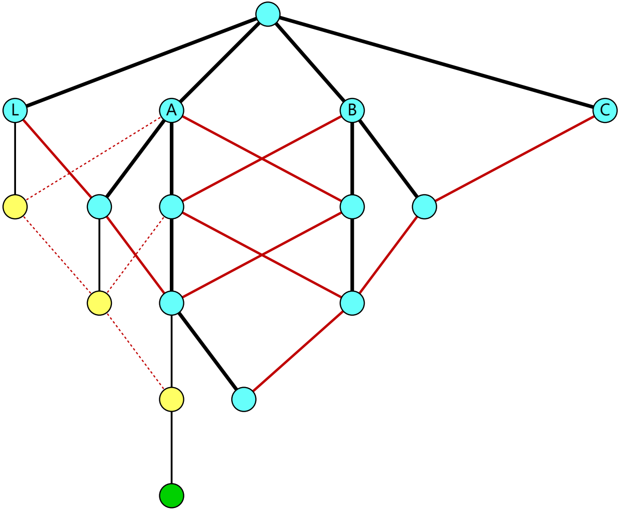

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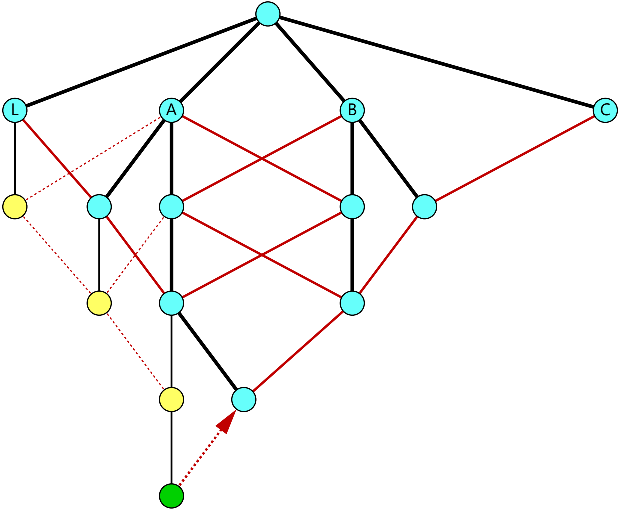

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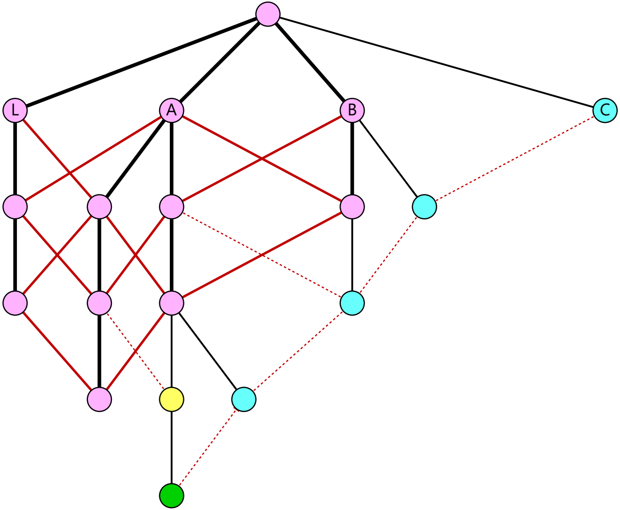

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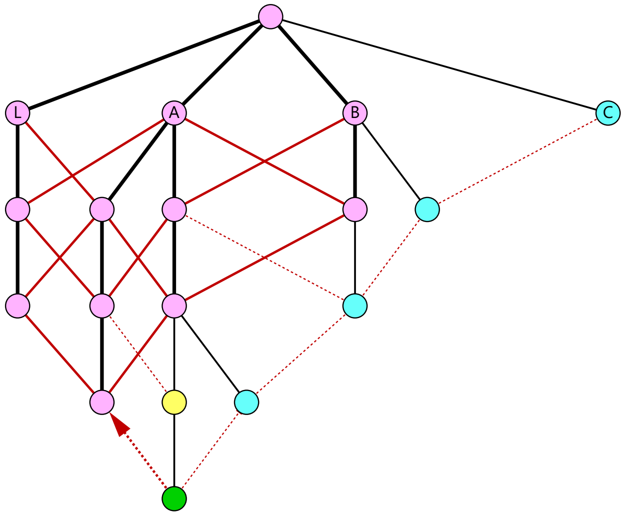

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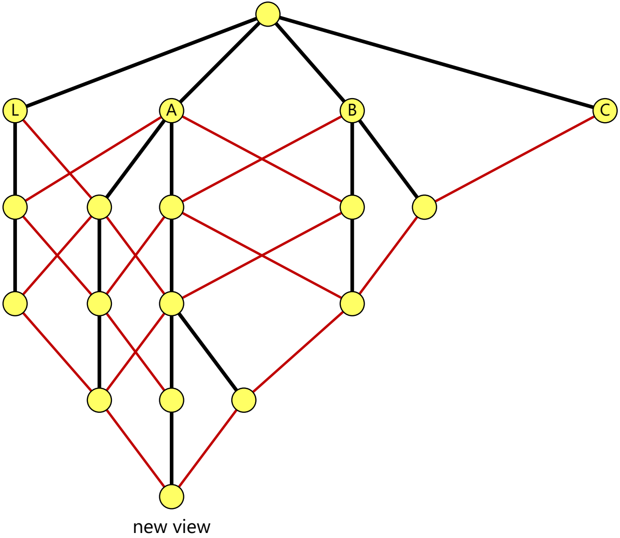


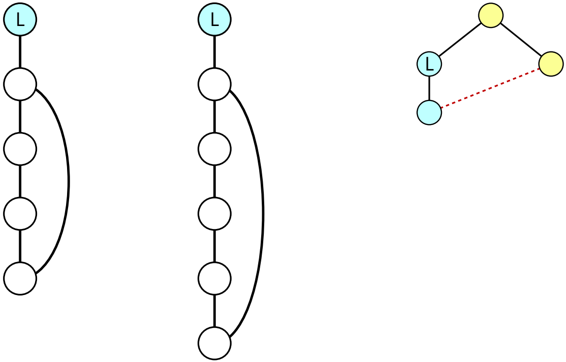
Lower Bound on Counting

Lower bound

Theorem

The Counting Problem is not solvable in less than $2n-3$ rounds.

System 1 System 2 Leaders' view



Round 1

Lower bound

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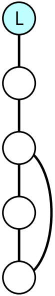
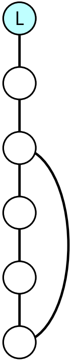


System 2



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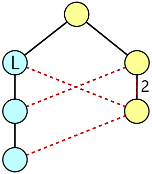

Round 2

Lower bound

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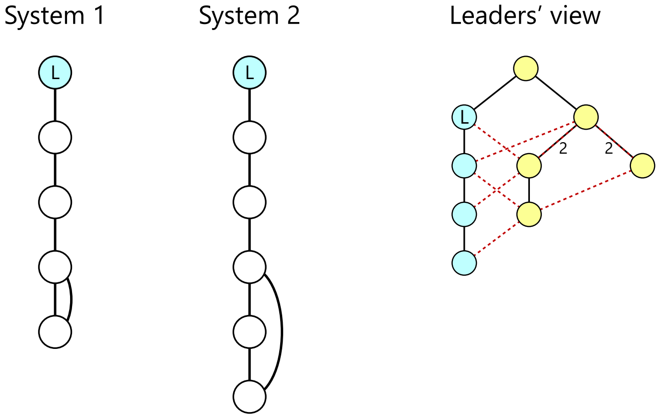

Round 3

Lower bound

Theorem

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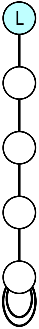
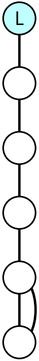


System 2



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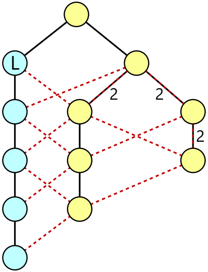

Round 4

Lower bound

Theorem

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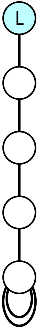
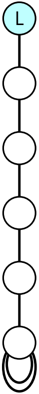
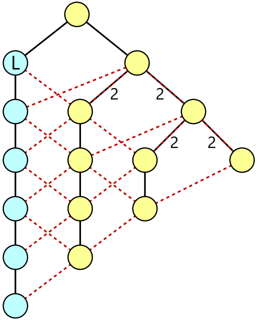


System 2



Leaders' view



Round 5

Lower bound

Theorem

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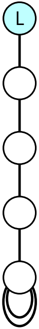
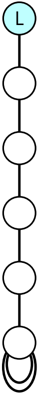
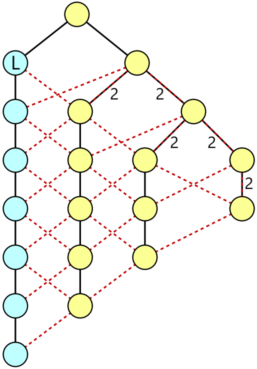


System 2



Leaders' view



Round 6

Lower bound

Theorem

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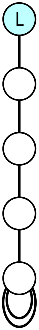
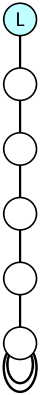
System 1



System 2



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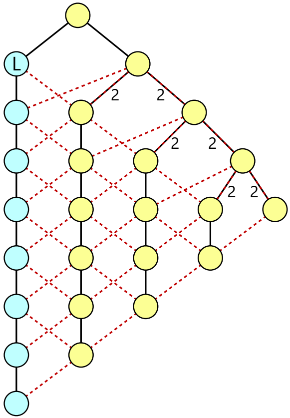

Round 7

Lower bound

Theorem

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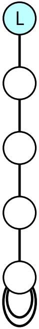
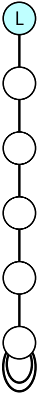


System 2



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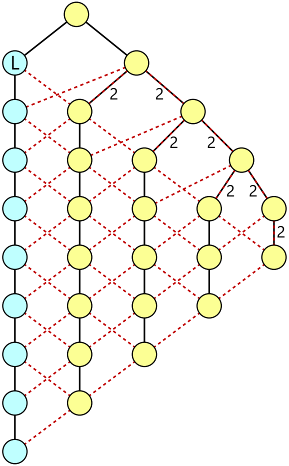
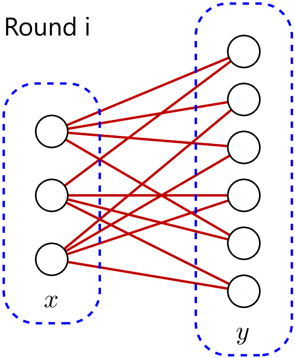


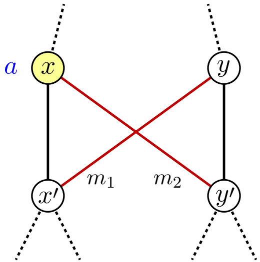
Round 8

Stabilizing Algorithm

Computing anonymities

Suppose we know the anonymity of a node x with a single child x'.





If the agents represented by x have observed agents whose corresponding node y has only one child y', then we can compute the anonymity of y and y', as well.

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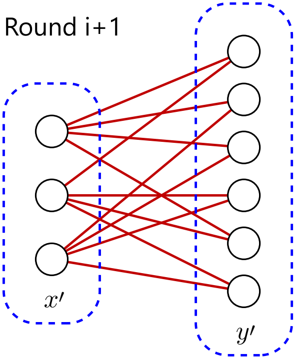
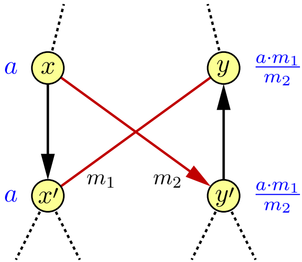
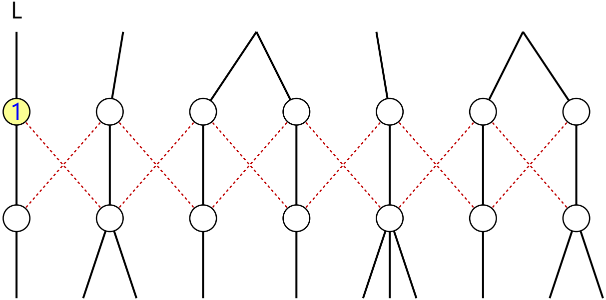




If the agents represented by x have observed agents whose corresponding node y has only one child y', then we can compute the anonymity of y and y', as well.

Stabilizing algorithm

If all nodes in a level have only one child, we can compute the anonymity of all of them (because the network is connected).

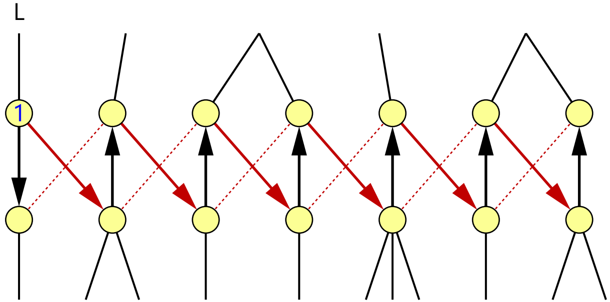


Since there are n agents, the tree can branch at most $n-1$ times.

Thus, among the first $n-1$ levels, there must be a level where no node branches. In this level, we can compute all anonymities.

Stabilizing algorithm

If all nodes in a level have only one child, we can compute the anonymity of all of them (because the network is connected).



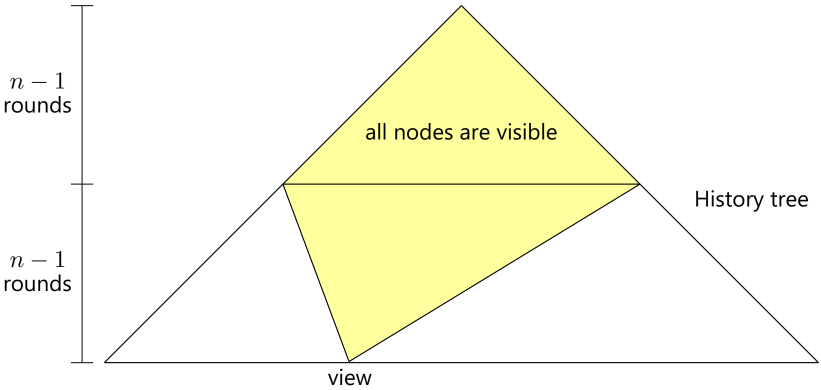
Since there are n agents, the tree can branch at most $n-1$ times.

Thus, among the first $n-1$ levels, there must be a level where no node branches. In this level, we can compute all anonymities.

Stabilizing algorithm

Theorem

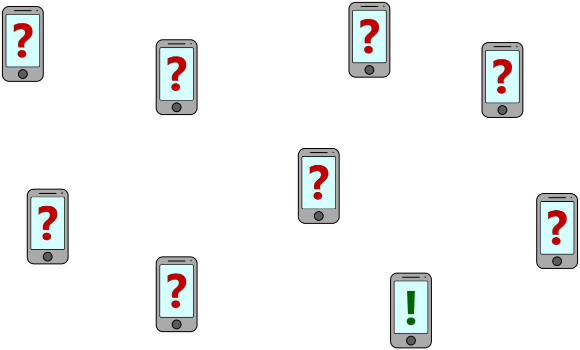
The Generalized Counting Problem can be stably solved in $2n-2$ rounds (without explicit termination).



Note that, after $2n-2$ rounds, all nodes in the first $n-1$ levels of the history tree are in the views of all agents.

Propagation of information

If the network is connected at all rounds, every news reaches every agent in at most $n-1$ rounds.



Hence, whenever two agents interact, all agents will know it within $n-1$ rounds (and it will show in their views of the history tree).

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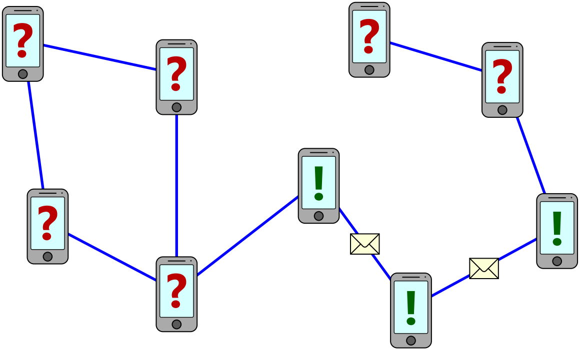

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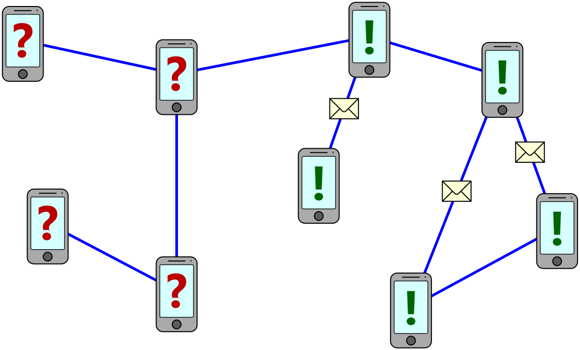

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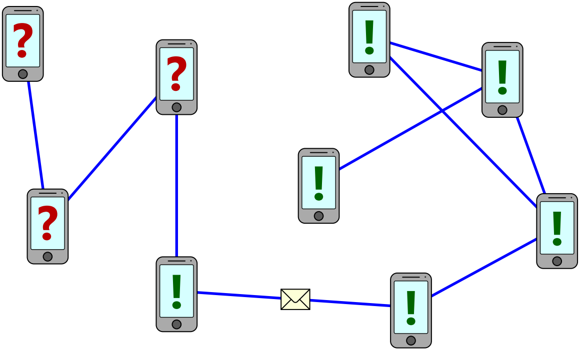

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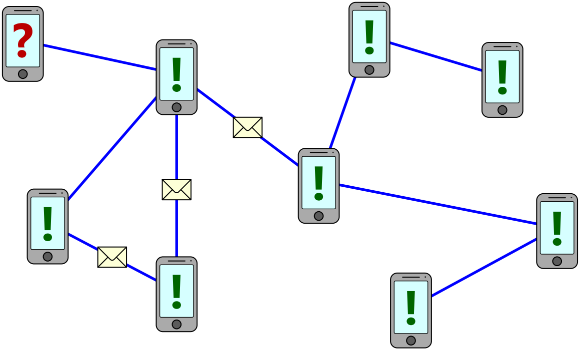

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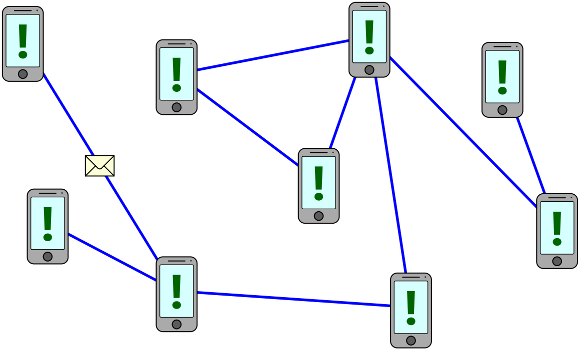
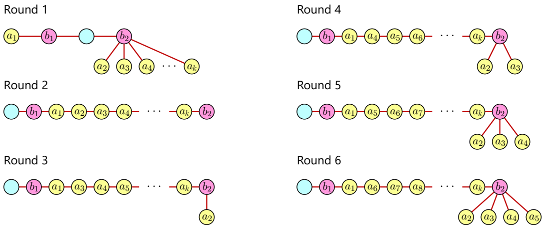
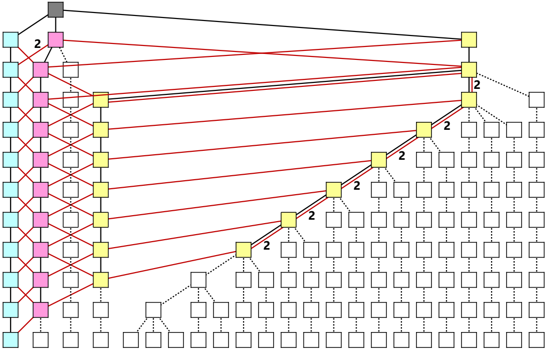


Hence, whenever two agents interact, all agents will know it within $n-1$ rounds (and it will show in their views of the history tree).

Terminating Algorithm

Termination needs a new technique





Terminating algorithm: Overview

We will give a *terminating* algorithm for the Counting Problem.

The algorithm is as follows:

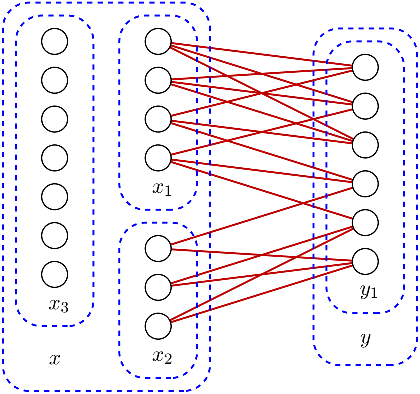
- Use the Leader's observations to make guesses on anonymities.
- In any set of n guesses, we can always identify a correct one.
- Once we have identified $n - 1$ correct guesses, we can use some of them to make new guesses on anonymities.
- Repeat until we have the anonymity of all visible branches of the history tree: this gives an estimate n' on n.
- Wait n' rounds to confirm the estimate; if correct, terminate.

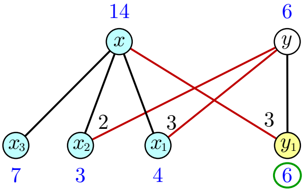
Theorem

The Generalized Counting Problem can be solved in $3n - 2$ rounds with explicit termination.

Guessing anonymities

Suppose we know the anonymities of a node x and its children. If some of the agents represented by x have observed agents represented by y, we can *guess* the anonymity of a child of y.





Guess on y_1: $\dfrac{4 \cdot 3 + 3 \cdot 2}{3} = 6$

If only one child of y has seen x, then the guess is *correct*.

Guessing anonymities

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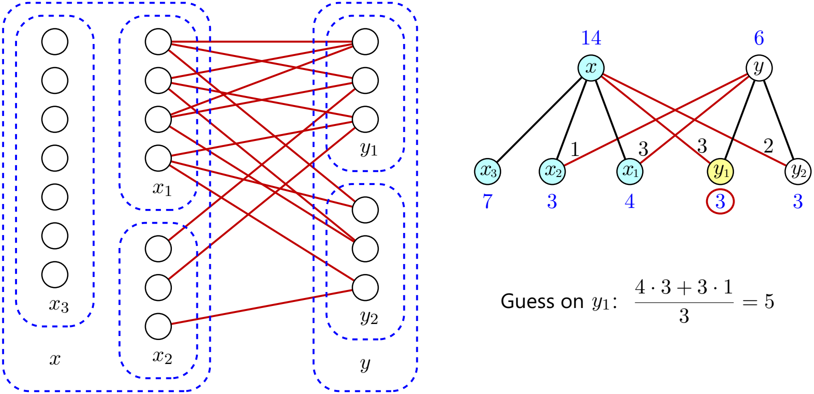
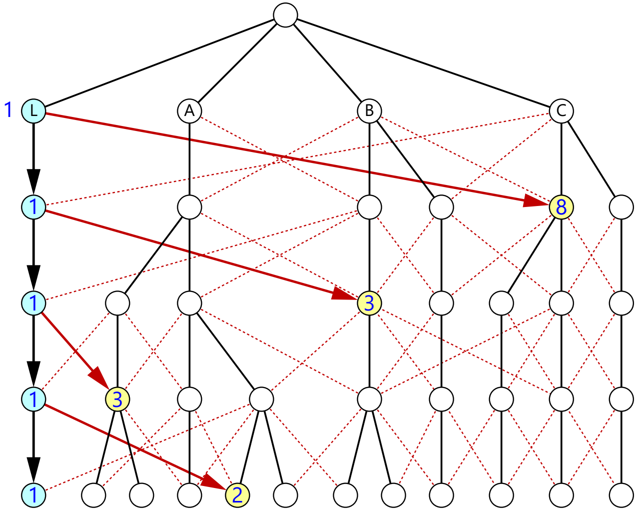


Guess on y_1: $\dfrac{4 \cdot 3 + 3 \cdot 1}{3} = 5$

Otherwise, the guess is an *overestimation* of the anonymity.

Guessing anonymities from the Leader

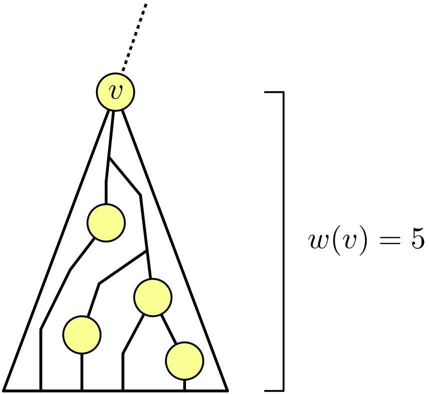
We can make one guess per round using the Leader's observations.



How do we know which guesses are correct?

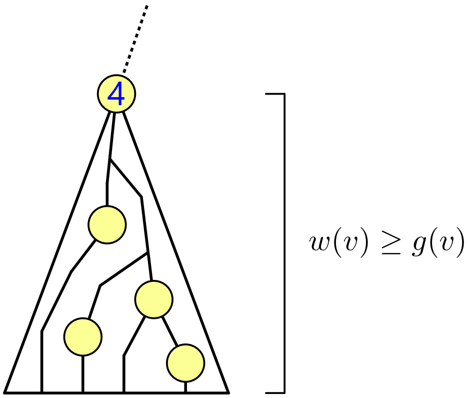
Weight of a node

When a node v has a guess, we define its *weight* $w(v)$ as the number of nodes in the subtree hanging from v that have guesses.



Weight of a node

A node v is *heavy* if its weight $w(v)$ is at least as large as the value of its guess $g(v)$.



Limiting theorem

We denote by $a(v)$ the anonymity of a node v, by $g(v)$ a guess on $a(v)$, and by $w(v)$ the weight of v.

Theorem

If all guesses are on different rounds and $w(v) > a(v)$, then some descendants of v are heavy.

Proof. By well-founded induction on $w(v)$.

Let $v_1, v_2, \ldots$ be the closest descendants of v that have guesses. Of course, $a(v) \geq \sum_i a(v_i)$.

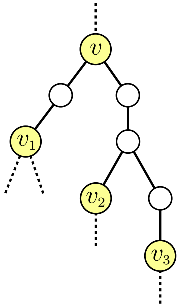
By the inductive hypothesis, $w(v_i) \leq a(v_i)$ for all i.

$w(v){-}1 = \sum_i w(v_i) \leq \sum_i a(v_i) \leq a(v) \leq w(v){-}1$

Thus, $w(v_i) = a(v_i)$ and $a(v) = \sum_i a(v_i)$.

The deepest node v_d has no siblings, because all guesses are on different rounds.

Hence $g(v_d) = a(v_d) = w(v_d)$, and v_d is heavy. □



Criterion of correctness

Corollary

If v is heavy and no descendant of v is heavy, then $g(v) = a(v)$.

Proof. By assumption, $g(v) \leq w(v)$.

By the limiting theorem, $w(v) \leq a(v)$.

Guesses never underestimate anonymities, and so $a(v) \leq g(v)$.

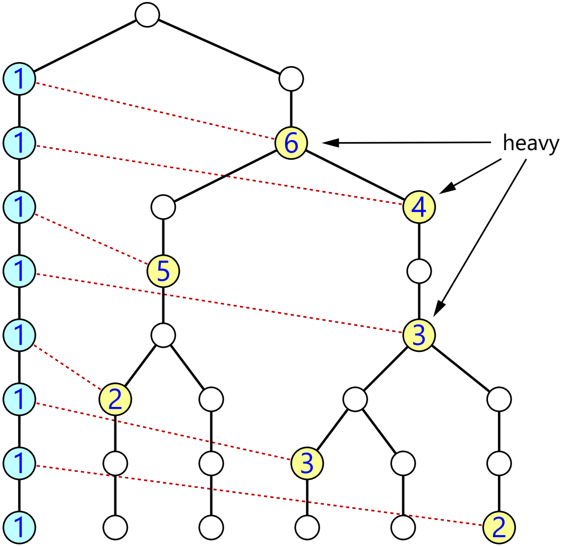
$g(v) \leq w(v) \leq a(v) \leq g(v)$, hence $g(v) = a(v)$. □

This corollary gives agents a criterion to determine when a guess is necessarily correct: **If v is heavy and no descendants of v are heavy, then the guess on v is correct.**

Moreover, by the limiting theorem, such a node v is found by the time there are n guesses in total.

Criterion of correctness: Example

Any agent with this view is able to determine which guess is necessarily correct:



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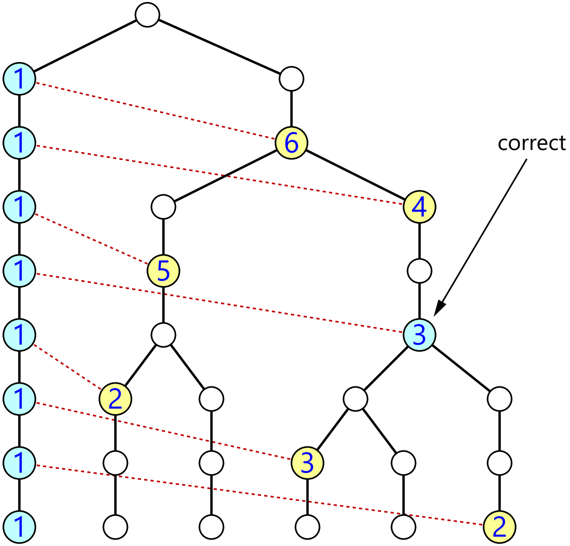
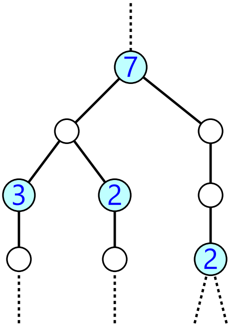


Propagation of guesses

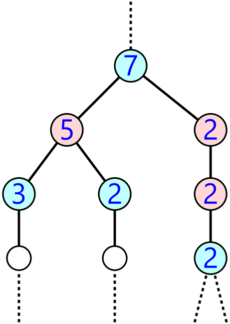
An *island* is a connected component of (a view of) the history tree that contains no leaves and does not contain the root.



Suppose that the nodes with necessarily correct guesses bound an island in the history tree.

Propagation of guesses

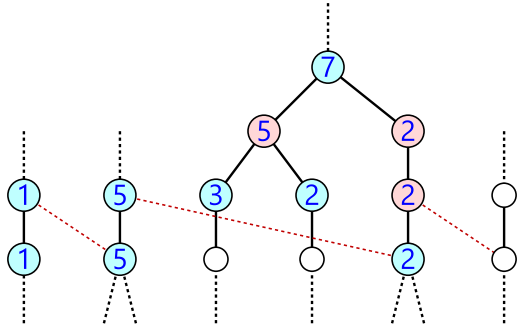
An *island* is a connected component of (a view of) the history tree that contains no leaves and does not contain the root.



If the anonymity of the top node is the sum of the bottom ones, then we can infer the anonymities of all the nodes in the island.

Propagation of guesses

An *island* is a connected component of (a view of) the history tree that contains no leaves and does not contain the root.



Since the network is connected at every round, we can make a new guess from one of the nodes in the island.

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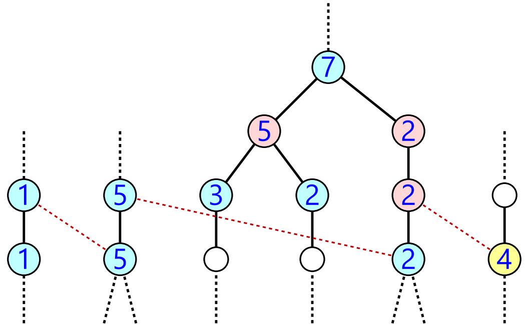

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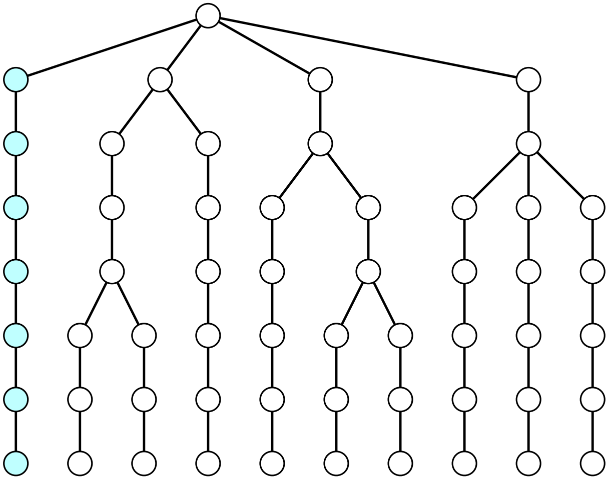
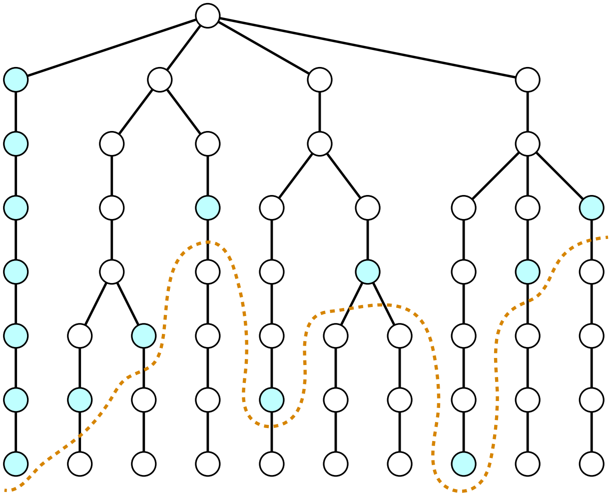


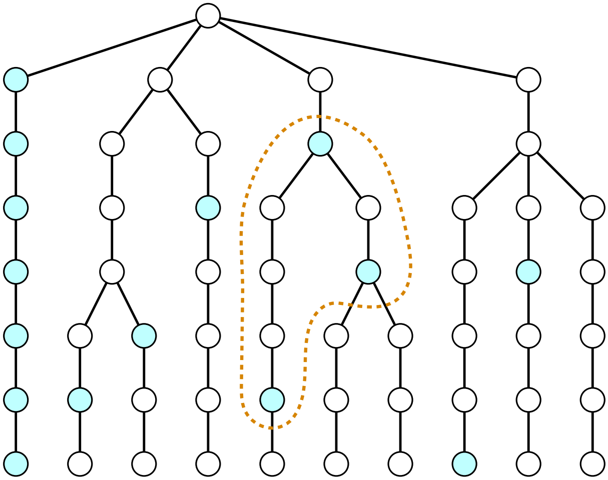
Suppose that there are $n-1$ nodes with necessarily correct guesses (other than the Leader ones). There are two cases:

Propagation of guesses



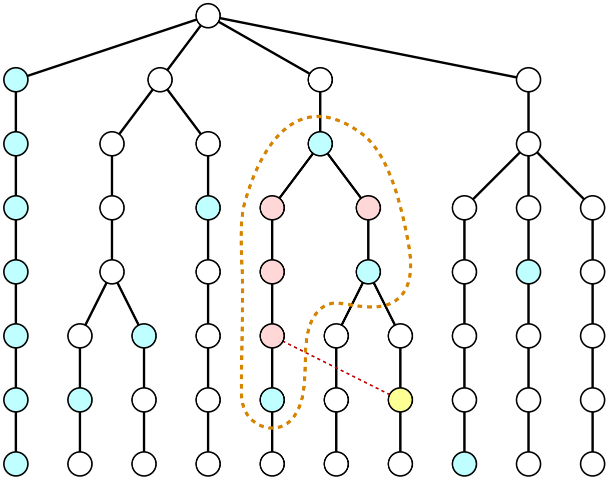
Either these nodes determine a *counting cut* of the history tree, in which case we have an estimate n' on n, given by their sum...

Propagation of guesses



...Or else, some of these nodes determine an island, which allows us to make a new guess, and so on.

Propagation of guesses

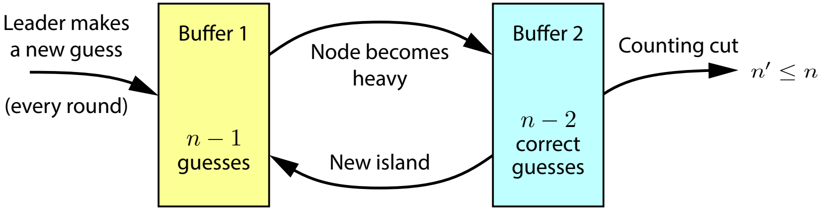


...Or else, some of these nodes determine an island, which allows us to make a new guess, and so on.

Dynamics of new guesses

Summarizing, there are two "buffers":

- A buffer of $n-1$ overestimating guesses (yellow nodes),
- A buffer of $n-2$ necessarily correct guesses (blue nodes).

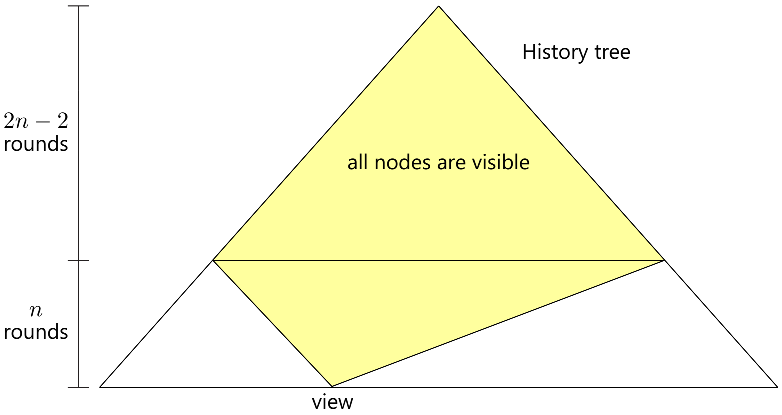


When both buffers are full, the chain of guesses "snowballs" and eventually produces a *counting cut*, which in turn yields an *estimate* $n' \leq n$. That happens within $2n-2$ rounds.

Termination condition

Once we have a cut and an estimate $n' \leq n$, we wait n' rounds.

If $n' < n$, a new node appears in the first levels of the history tree.



If $n' = n$, then no new nodes appear, and the algorithm terminates.

Further results and open problems

For the Counting problem, we have:

- A terminating algorithm in $3n$ rounds;
- A lower bound of $2n$ rounds.

Open problem

Can we close the gap between $2n$ and $3n$?

We can extend our results in two directions:

- For networks with $\ell > 0$ **Leaders**, we have a terminating algorithm in $(\ell^2 + \ell + 1)n$ rounds.
- For **congested networks**, we have a terminating algorithm in $O(n^3)$ rounds with message size limited to $O(\log n)$.

Open problem

What if we have $\ell > 1$ Leaders and $O(\log n)$-size messages?