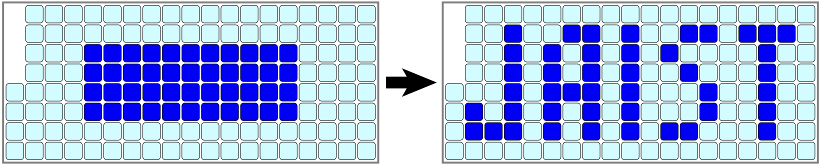
Compaction and Gravity Puzzles

Giovanni Viglietta

Joint work with Hugo Akitaya and Maarten Löffler

JAIST – February 7, 2022

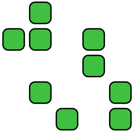


Overview

- Compaction puzzle
  - Introduction
  - Previous work
  - Sparse configurations
- Gravity puzzle
  - Introduction
  - Only even permutations are possible
  - All even permutations are possible
- Open problems

Compaction puzzle

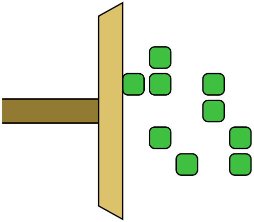
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

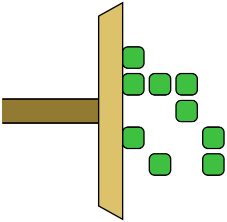
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

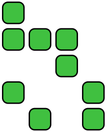
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

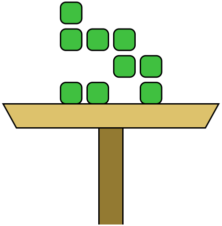
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

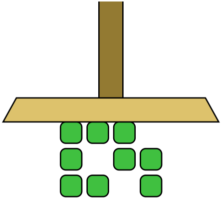
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

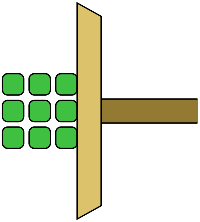
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

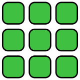
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Compaction puzzle

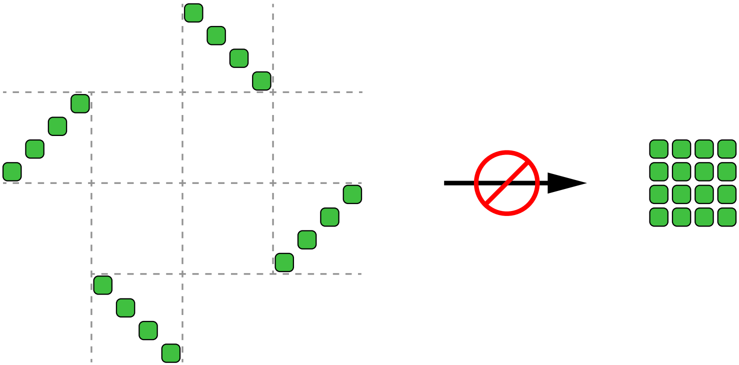
There are 9 tokens on a grid. Can we reconfigure them into a square box by pushing them from the sides?



Problem: Given n tokens, can we push them into an $a \times b$ box?

Sparse configurations

What if the configuration is *"sparse"*, i.e., no row or column has more than one token? Forming a square may still be impossible:



Compaction puzzle: Previous results

Essentially one paper:

H. Akitaya, G. Aloupis, M. Löffler, and A. Rounds
"Trash compaction", in *Proceedings of EuroCG 2016*

Theorem

Deciding if tokens can be pushed into an $a \times b$ box is NP-complete.

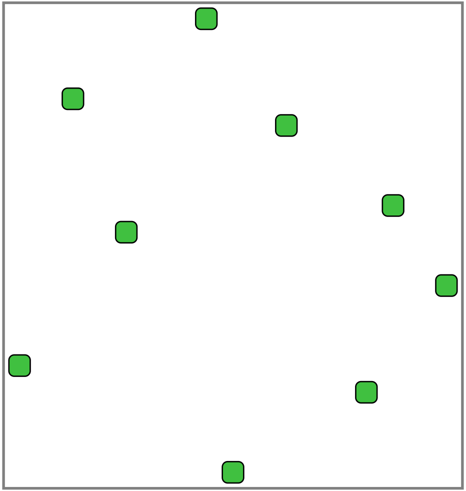
Theorem

Given $2n$ tokens that occupy k rows, deciding if they can be pushed into an $n \times 2$ box is an XP problem (with parameter k). Specifically, it can be solved in $O(n)$ time if $k \leq 3$ and in $O(n^{2(k-2)})$ time if $k > 3$.

Open problem

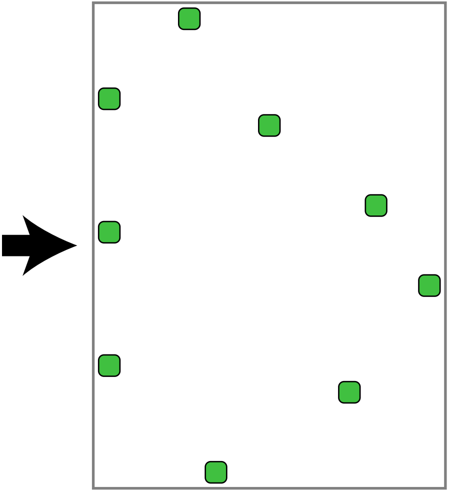
Is there a sparse configuration of $n = 9$ tokens that cannot be pushed into a 3×3 square?

Pushing sparse configurations into boxes



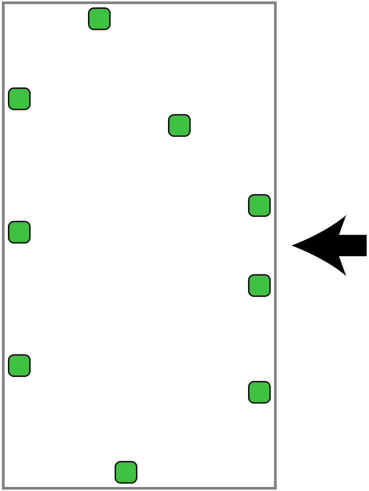
There is an algorithm for pushing 9 tokens into a 3×3 square.

Pushing sparse configurations into boxes



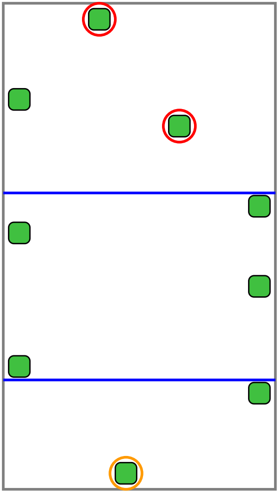
Step 1: Push right until 3 “side tokens” become aligned.

Pushing sparse configurations into boxes



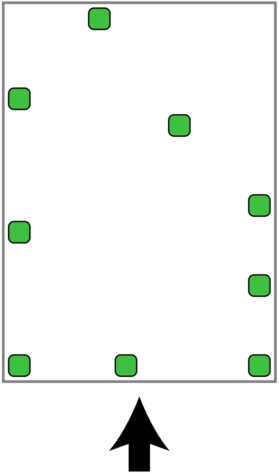
Step 2: Push left until 3 “side tokens” become aligned.

Pushing sparse configurations into boxes



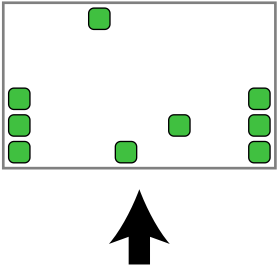
Identify the side (up or down) with fewer “middle tokens”.

Pushing sparse configurations into boxes



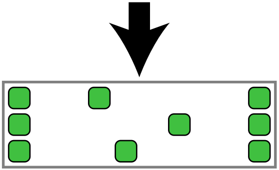
Step 3: Push from that side until 3 tokens become aligned.

Pushing sparse configurations into boxes



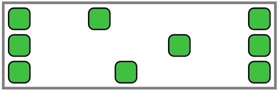
Step 4: Push up/down as far as possible without aligning 4 tokens.

Pushing sparse configurations into boxes



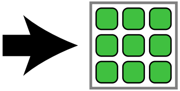
Step 4: Push up/down as far as possible without aligning 4 tokens.

Pushing sparse configurations into boxes



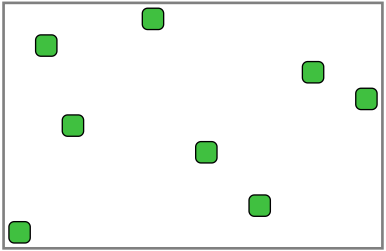
Now the tokens occupy exactly 3 consecutive rows.

Pushing sparse configurations into boxes



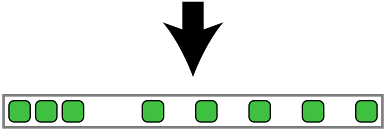
Step 5: Push all the way to the left to form a 3×3 square.

Pushing sparse configurations into boxes



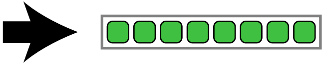
Forming an $n \times 1$ box is easy: simply push down and then right.

Pushing sparse configurations into boxes



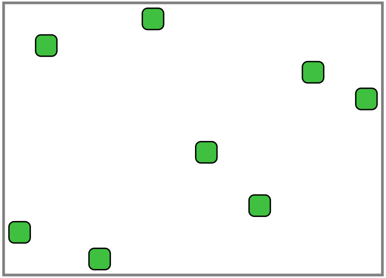
Forming an $n \times 1$ box is easy: simply push down and then right.

Pushing sparse configurations into boxes



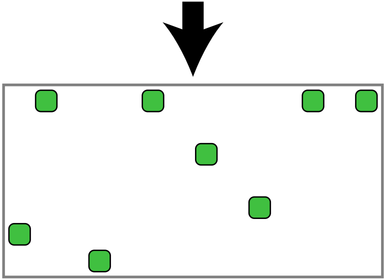
Forming an $n \times 1$ box is easy: simply push down and then right.

Pushing sparse configurations into boxes



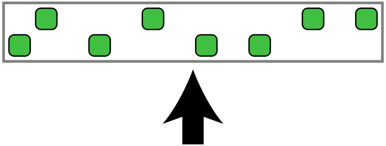
Given $2n$ tokens, forming an $n \times 2$ box is easy, too.

Pushing sparse configurations into boxes



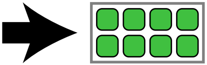
Given $2n$ tokens, forming an $n \times 2$ box is easy, too.

Pushing sparse configurations into boxes



Given $2n$ tokens, forming an $n \times 2$ box is easy, too.

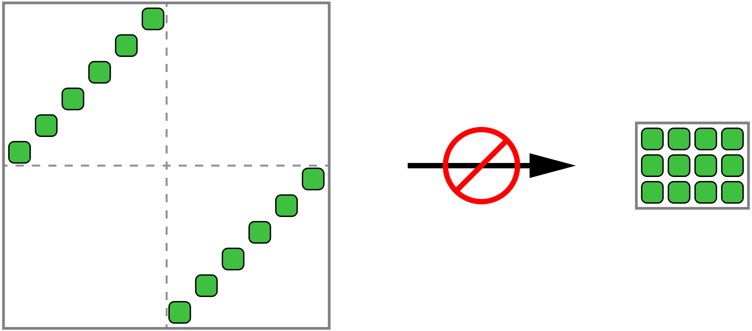
Pushing sparse configurations into boxes



Given $2n$ tokens, forming an $n \times 2$ box is easy, too.

Pushing sparse configurations into boxes

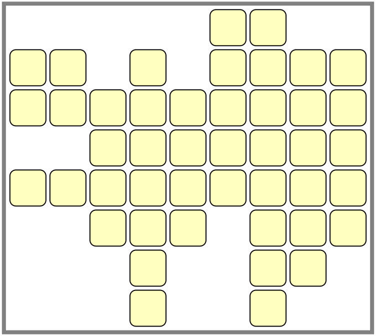
In all other cases, i.e., when $a \geq 4$ and $b \geq 3$, there are sparse configurations of $n = ab$ tokens that cannot form an $a \times b$ box:



Theorem

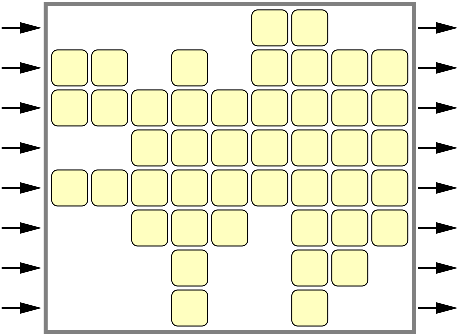
All sparse configurations of $n = ab$ tokens can be pushed into an $a \times b$ box if and only if $a \leq 2$ or $b \leq 2$ or $a = b = 3$.

Uncompressible configurations



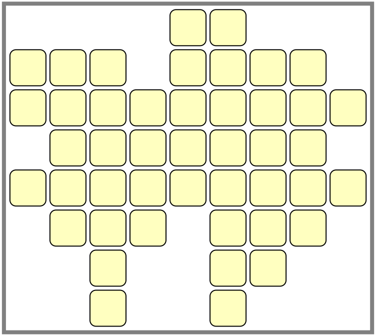
From now on, we will focus on *uncompressible* configurations, i.e., where the bounding box contains a full row and a full column.

Uncompressible configurations



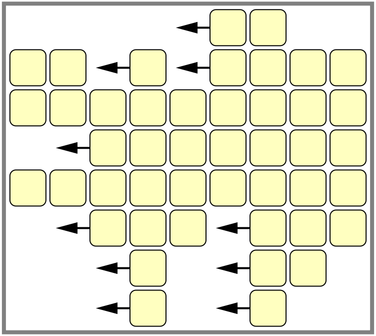
In such configurations, pushing the bounding box in one direction is equivalent to letting tokens “fall” in the opposite direction.

Uncompressible configurations



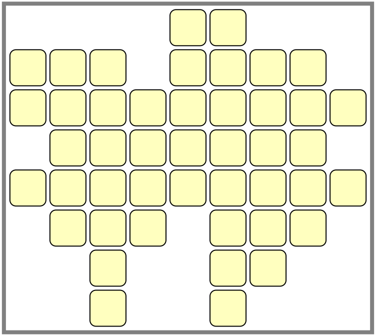
In such configurations, pushing the bounding box in one direction is equivalent to letting tokens “fall” in the opposite direction.

Uncompressible configurations



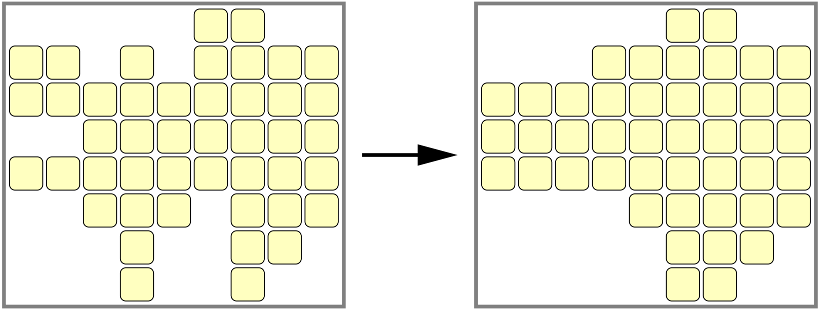
In such configurations, pushing the bounding box in one direction is equivalent to letting tokens "fall" in the opposite direction.

Uncompressible configurations



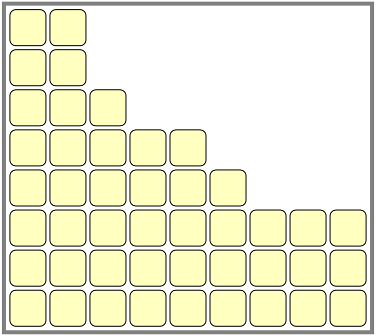
In such configurations, pushing the bounding box in one direction is equivalent to letting tokens "fall" in the opposite direction.

Uncompressible configurations



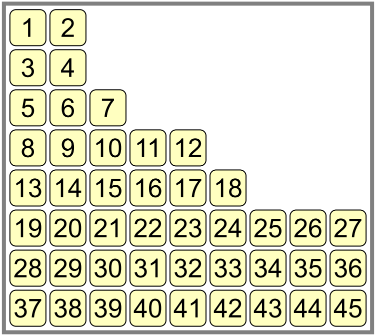
After a few moves, the configuration tends to become orthogonally convex, where each row (resp. column) of length k is contained in the projection of every row (resp. column) of length $\geq k$.
We call such a configuration *compact*.

Uncompressible configurations



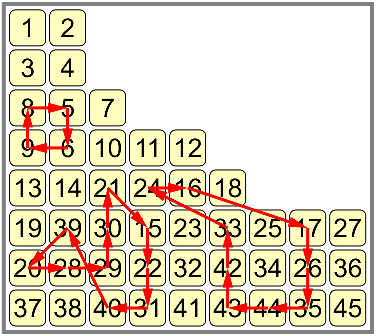
Without loss of generality, we will assume the configuration to be a compact “staircase” with a full leftmost column and a full bottommost column. We call such a configuration *canonical*.

Gravity puzzle



Let us assign unique labels to the tokens. After playing some moves and restoring a canonical configuration, we obtain a permutation of these tokens.

Gravity puzzle

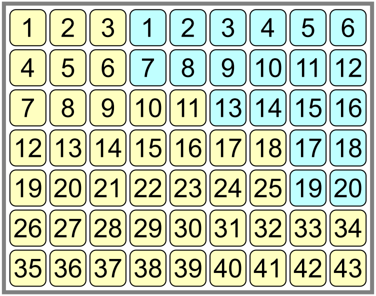


Problem: What are the possible configurations of this puzzle?

Note: The set of possible permutations is closed under composition, and therefore is a *permutation group*.*

*For *finite* sets of permutations, closure under composition is sufficient.

Gravity puzzle: Permutations are even



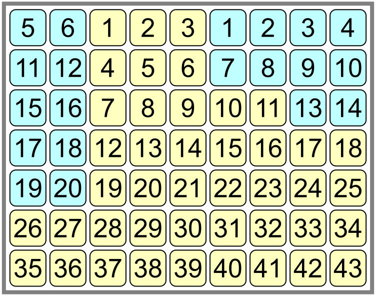
We will prove that any permutation obtainable in this puzzle must be <u>even</u>. It will be convenient to label the empty cells, too.

Gravity puzzle: Permutations are even

6	1	2	3	1	2	3	4	5
12	4	5	6	7	8	9	10	11
16	7	8	9	10	11	13	14	15
18	12	13	14	15	16	17	18	17
20	19	20	21	22	23	24	25	19
26	27	28	29	30	31	32	33	34
35	36	37	38	39	40	41	42	43

Every move causes a cyclic permutation on some rows or columns, involving both labeled tokens and labeled empty cells.

Gravity puzzle: Permutations are even



Every move causes a cyclic permutation on some rows or columns, involving both labeled tokens and labeled empty cells.

Gravity puzzle: Permutations are even

4	5	6	1	2	3	1	2	3
10	11	12	4	5	6	7	8	9
14	15	16	7	8	9	10	11	13
17	18	12	13	14	15	16	17	18
19	20	19	20	21	22	23	24	25
26	27	28	29	30	31	32	33	34
35	36	37	38	39	40	41	42	43

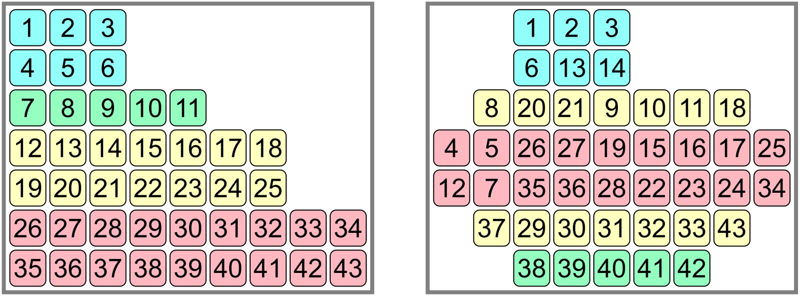
Every move causes a cyclic permutation on some rows or columns, involving both labeled tokens and labeled empty cells.

Gravity puzzle: Permutations are even

10	11	12	1	2	3	7	8	9
14	15	16	4	5	6	10	11	13
17	18	12	7	8	9	16	17	18
19	20	19	13	14	15	23	24	25
26	27	28	20	21	22	32	33	34
35	36	37	29	30	31	41	42	43
4	5	6	38	39	40	1	2	3

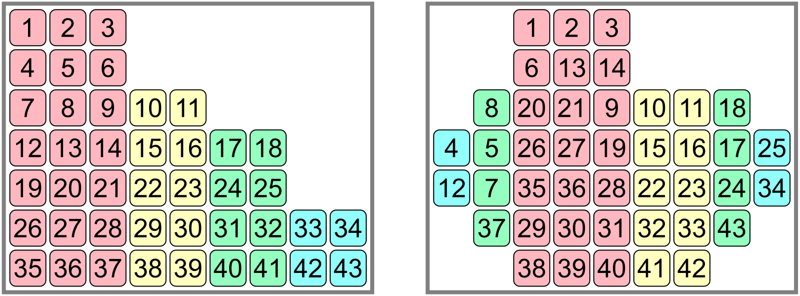
Every move causes a cyclic permutation on some rows or columns, involving both labeled tokens and labeled empty cells.

Gravity puzzle: Permutations are even



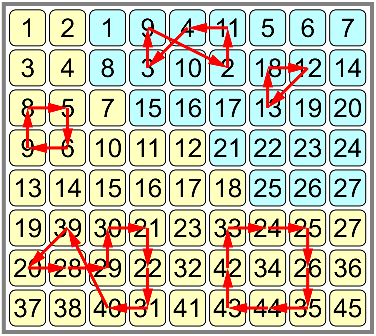
Recall that all rows of the same length must always be aligned. Moreover, for every move that pushes all rows of some length to the right, there must be a move that pushes all of them to the left.

Gravity puzzle: Permutations are even



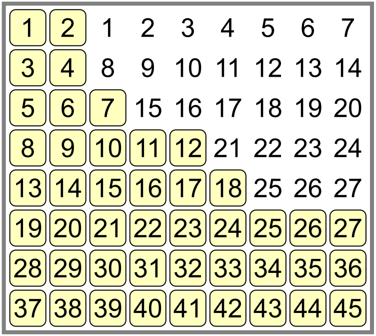
The same holds for columns and up/down moves. Thus, if we decompose every move into cycles (involving both tokens and empty cells), we see that every cycle must have a matching cycle of the same length.

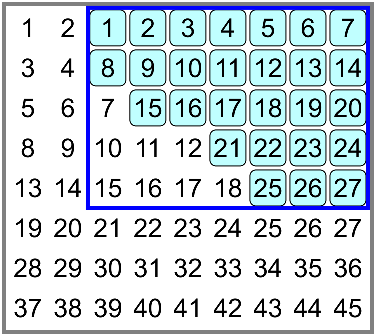
Gravity puzzle: Permutations are even



So, if we restore a canonical configuration, the overall permutation must be <u>even</u>. We still need to prove that the same permutation, restricted to the tokens (equiv., to the empty cells), is also even.

Gravity puzzle: Primal and dual puzzles





Let us isolate the empty cells, and treat them as a "dual puzzle".

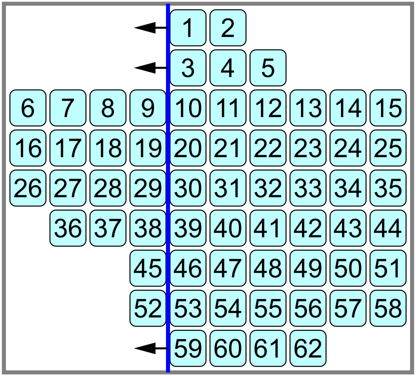
Gravity puzzle: Primal and dual puzzles

16	21	22	23	24	7	11	8	15
25	26	27	1	2	10	16	12	18
8	9	21	3	4	15	25	17	27
13	14	30	5	6	24	34	26	36
19	20	39	22	23	32	33	35	45
4	5	6	7	31	40	41	43	44
3	11	12	13	14	28	29	42	2
10	17	18	19	20	37	38	1	9

40	41	43	44	4	5	6	7	31
28	29	42	2	3	11	12	13	14
37	38	1	9	10	17	18	19	20
7	11	8	15	16	21	22	23	24
10	16	12	18	25	26	27	1	2
15	25	17	27	8	9	21	3	4
24	34	26	36	13	14	30	5	6
32	33	35	45	19	20	39	22	23

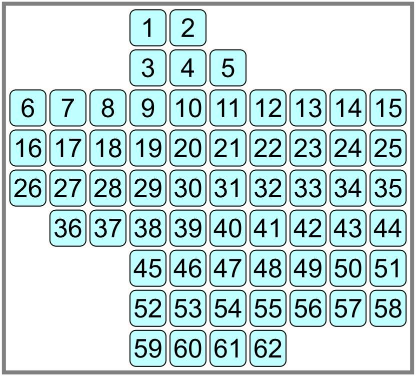
As we play the primal puzzle, we are also playing the dual puzzle.

Gravity puzzle: Primal and dual puzzles



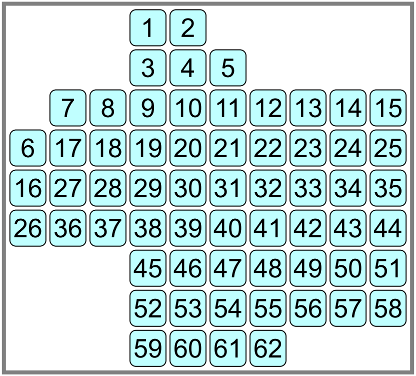
The rules of the dual puzzle as slightly different: in a left move, only the rows that are farthest from the left margin are pushed, etc.

Gravity puzzle: Primal and dual puzzles



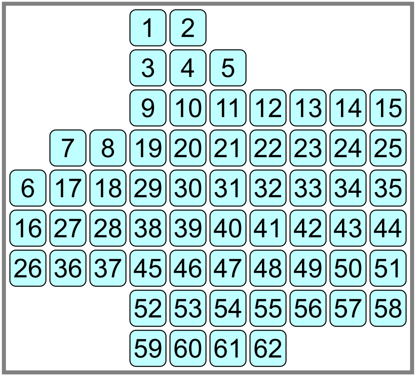
The rules of the dual puzzle as slightly different: in a left move, only the rows that are farthest from the left margin are pushed, etc.

Gravity puzzle: Primal and dual puzzles



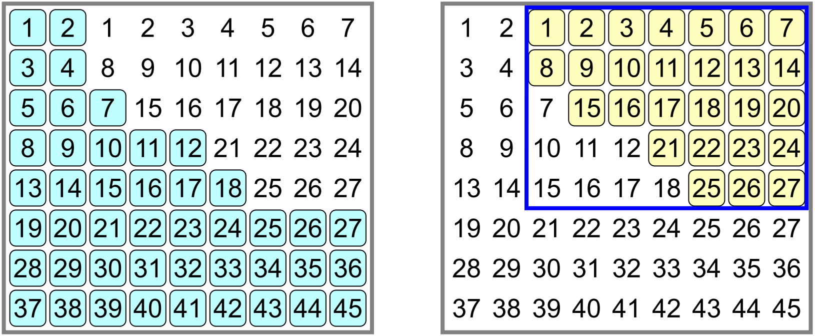
The rules of the dual puzzle as slightly different: in a left move, only the rows that are farthest from the left margin are pushed, etc.

Gravity puzzle: Primal and dual puzzles



The rules of the dual puzzle as slightly different: in a left move, only the rows that are farthest from the left margin are pushed, etc.

Gravity puzzle: Primal and dual puzzles



The dual puzzle of a dual-type puzzle is again a primal-type puzzle.

7	1	2	1	2	3	4	5	6
14	3	4	8	9	10	11	12	13
5	6	7	15	16	17	18	19	20
8	9	10	11	12	21	22	23	24
13	14	15	16	17	18	25	26	27
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45

1	2	1	2	3	4	5	6	7
3	4	8	9	10	11	12	13	14
6	7	15	16	17	18	19	20	5
9	10	11	12	21	22	23	24	8
14	15	16	17	18	25	26	27	13
20	21	22	23	24	25	26	27	19
29	30	31	32	33	34	35	36	28
38	39	40	41	42	43	44	45	37

The dual puzzle of a dual-type puzzle is again a primal-type puzzle.

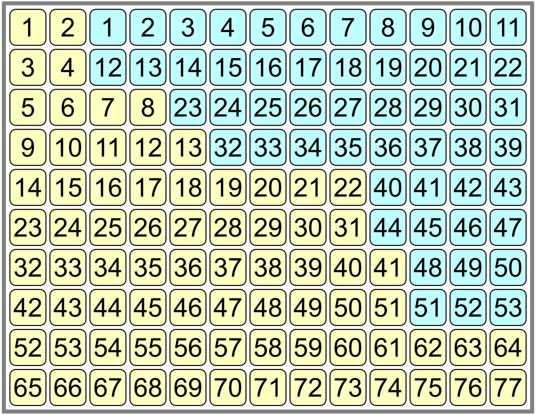
Gravity puzzle: Primal and dual puzzles

7	1	2	1	2	3	11	12	13
14	3	4	8	9	10	18	19	20
5	6	7	15	16	17	22	23	24
8	9	10	11	12	21	25	26	27
13	14	15	16	17	18	25	26	27
19	20	21	22	23	24	34	35	36
28	29	30	31	32	33	43	44	45
37	38	39	40	41	42	4	5	6

38	39	40	41	42	4	5	6	37
1	2	1	2	3	11	12	13	7
3	4	8	9	10	18	19	20	14
6	7	15	16	17	22	23	24	5
9	10	11	12	21	25	26	27	8
14	15	16	17	18	25	26	27	13
20	21	22	23	24	34	35	36	19
29	30	31	32	33	43	44	45	28

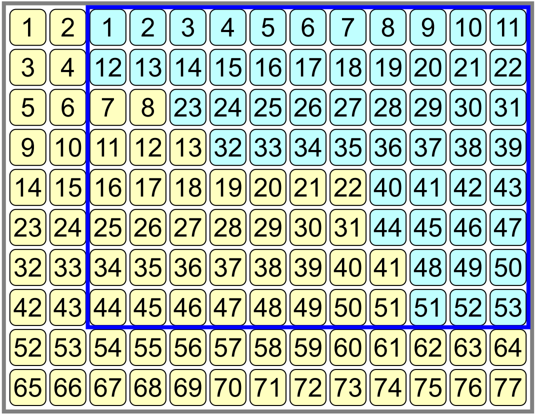
The dual puzzle of a dual-type puzzle is again a primal-type puzzle.

Gravity puzzle: Permutations are even



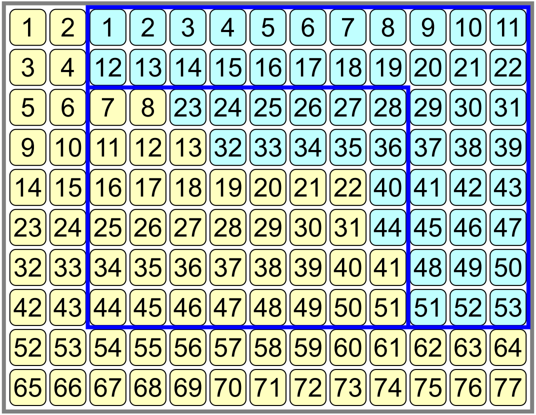
Since the dual puzzle is smaller than the primal, we can conclude by induction that the permutation restricted to each puzzle is even.

Gravity puzzle: Permutations are even



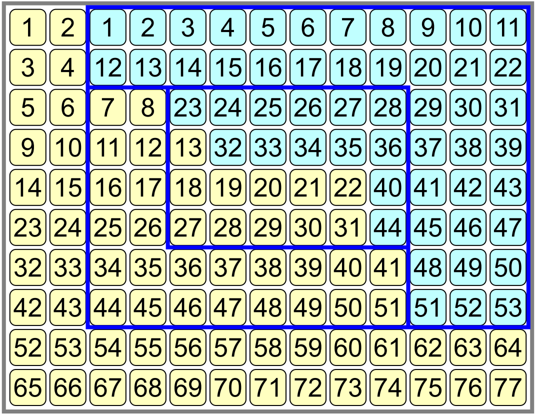
Since the dual puzzle is smaller than the primal, we can conclude by induction that the permutation restricted to each puzzle is even.

Gravity puzzle: Permutations are even



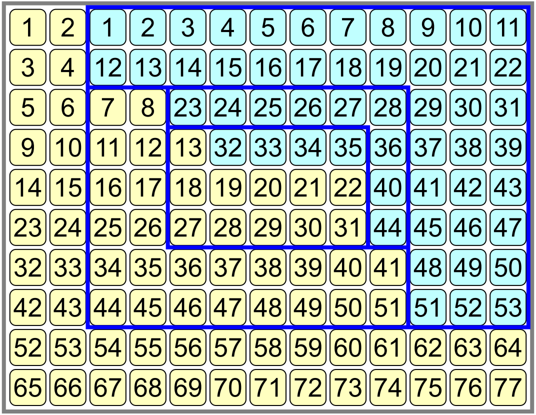
Since the dual puzzle is smaller than the primal, we can conclude by induction that the permutation restricted to each puzzle is even.

Gravity puzzle: Permutations are even



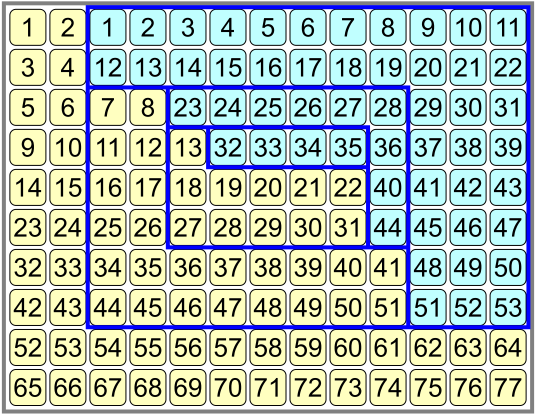
Since the dual puzzle is smaller than the primal, we can conclude by induction that the permutation restricted to each puzzle is even.

Gravity puzzle: Permutations are even



Since the dual puzzle is smaller than the primal, we can conclude by induction that the permutation restricted to each puzzle is even.

Gravity puzzle: Permutations are even



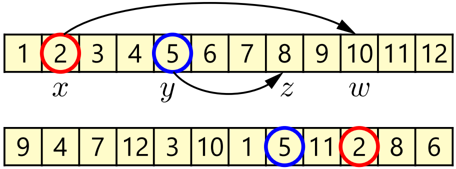
Since the dual puzzle is smaller than the primal, we can conclude by induction that the permutation restricted to each puzzle is even.

2-transitive groups

We have proved that only even permutations of the tokens are possible. Can we obtain *all* even permutations?

Definition

A permutation group G on $\{1, 2, \ldots, n\}$ is *2-transitive* if, for every $1 \le x, y, w, z \le n$ with $x \ne y$ and $w \ne z$, there is a permutation $\pi \in G$ such that $\pi(x) = w$ and $\pi(y) = z$.



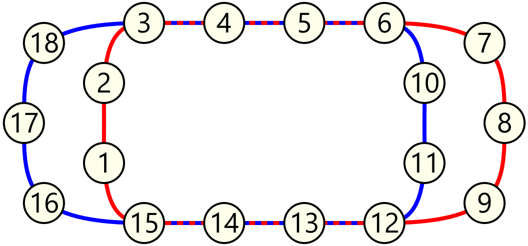
Theorem (Jones, 2014)

If a 2-transitive permutation group G on n items contains a cycle of length $n - 3$ or less, then G contains all even permutations.

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ *and* $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ *be two cycles spanning* $n = 4a+2b+2$ *items, with* $a \geq 0$ *and* $b \geq 1$. *Then,* α *and* β *generate a 2-transitive permutation group on the* n *items.*

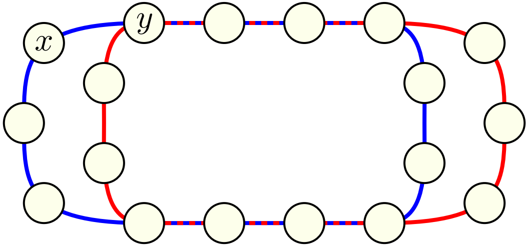


(Example with $a = 2$ and $b = 4$.)

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ *and* $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ *be two cycles spanning* $n = 4a+2b+2$ *items, with* $a \geq 0$ *and* $b \geq 1$*. Then,* α *and* β *generate a 2-transitive permutation group on the* n *items.*

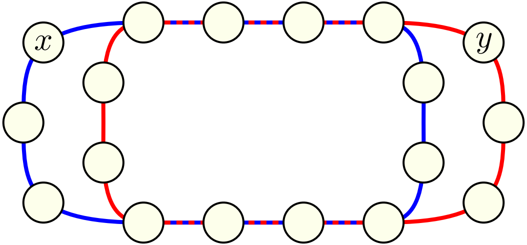


Assume WLOG that tokens x and y are adjacent along β.

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ *and* $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ *be two cycles spanning* $n = 4a+2b+2$ *items, with* $a \geq 0$ *and* $b \geq 1$*. Then,* α *and* β *generate a 2-transitive permutation group on the* n *items.*

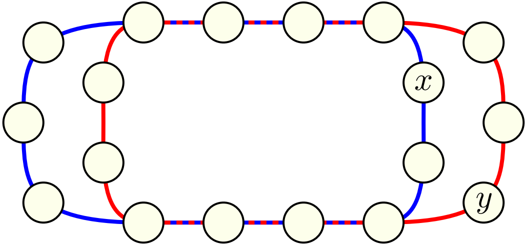


It is easy to put x and y on distinct cycles.

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ *and* $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ *be two cycles spanning* $n = 4a+2b+2$ *items, with* $a \geq 0$ *and* $b \geq 1$*. Then,* α *and* β *generate a 2-transitive permutation group on the* n *items.*

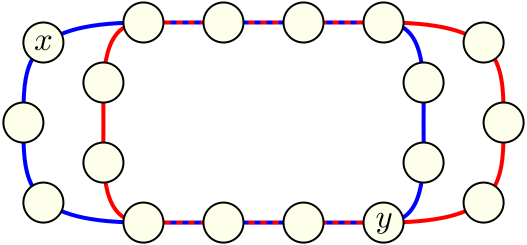


If their destinations are not on the same cycle, we simply shift them along each cycle the appropriate number of times.

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ and $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ be two cycles spanning $n = 4a+2b+2$ items, with $a \geq 0$ and $b \geq 1$. Then, α and β generate a 2-transitive permutation group on the n items.

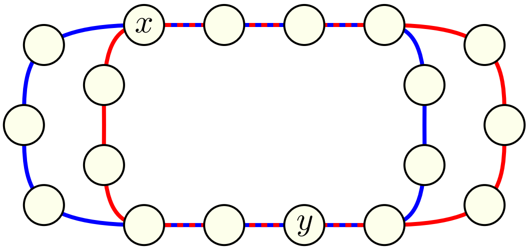


If their destinations are both on α, we put y at the appropriate distance along α, and then we shift x into α.

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ *and* $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ *be two cycles spanning* $n = 4a+2b+2$ *items, with* $a \geq 0$ *and* $b \geq 1$*. Then,* α *and* β *generate a 2-transitive permutation group on the* n *items.*

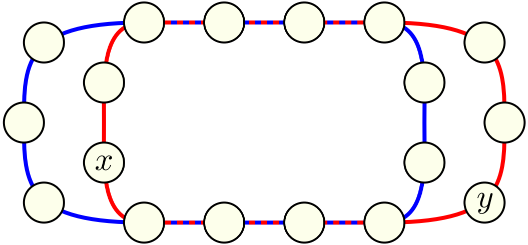


If their destinations are both on α, we put y at the appropriate distance along α, and then we shift x into α.

2-transitive groups

Lemma

Let $\alpha = (1, \ldots, 2a+b+1, 3a+b+2, \ldots, 3a+2b+1)$ *and* $\beta = (a+1, \ldots, a+b, 2a+b+2, \ldots, 4a+2b+2)$ *be two cycles spanning* $n = 4a+2b+2$ *items, with* $a \geq 0$ *and* $b \geq 1$*. Then,* α *and* β *generate a 2-transitive permutation group on the* n *items.*

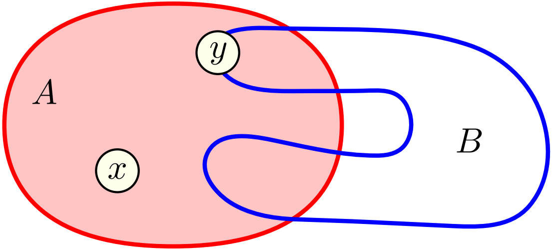


Finally, we shift x and y along α the appropriate number of times.

2-transitive groups

Lemma

Let G be a 2-transitive permutation group on a set A, and let β be a cycle spanning a set B, where $A \cap B \neq \emptyset$ and $A \setminus B \neq \emptyset$. Then, G and β generate a 2-transitive permutation group on $A \cup B$.

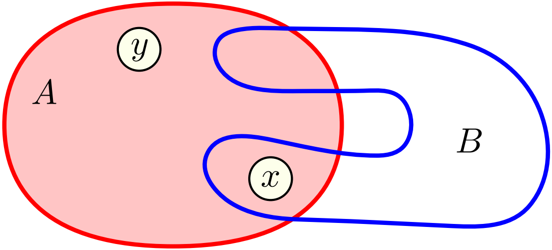


Assume WLOG that $x \in A \setminus B$ and $y \in A \cap B$.

2-transitive groups

Lemma

Let G be a 2-transitive permutation group on a set A, and let β be a cycle spanning a set B, where $A \cap B \neq \emptyset$ and $A \setminus B \neq \emptyset$. Then, G and β generate a 2-transitive permutation group on $A \cup B$.

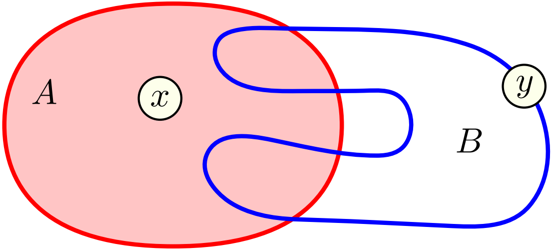


If their destinations are both in A, we move them via G.

2-transitive groups

Lemma

Let G be a 2-transitive permutation group on a set A, and let β be a cycle spanning a set B, where $A \cap B \neq \emptyset$ and $A \setminus B \neq \emptyset$. Then, G and β generate a 2-transitive permutation group on $A \cup B$.

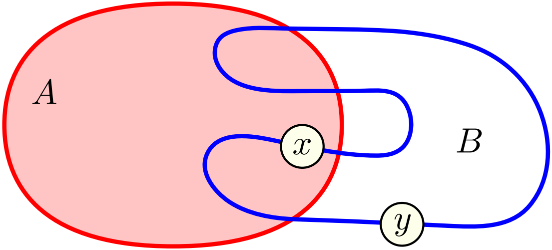


If the destination of x is in A and the destination of y is in $B \setminus A$, we first move y via β and then x via G.

2-transitive groups

Lemma

Let G be a 2-transitive permutation group on a set A, and let β be a cycle spanning a set B, where $A \cap B \neq \emptyset$ and $A \setminus B \neq \emptyset$. Then, G and β generate a 2-transitive permutation group on $A \cup B$.

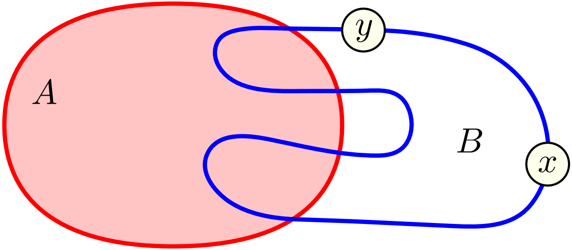


If both destinations are in $B \setminus A$, we first move y at the appropriate distance along β, and then we move x within A.

2-transitive groups

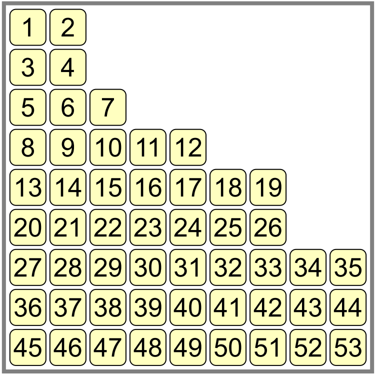
Lemma

Let G be a 2-transitive permutation group on a set A, and let β be a cycle spanning a set B, where $A \cap B \neq \emptyset$ and $A \setminus B \neq \emptyset$. Then, G and β generate a 2-transitive permutation group on $A \cup B$.



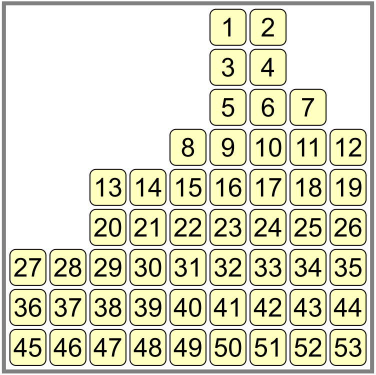
Finally, we shift x and y along β the appropriate number of times.

Gravity puzzle: Generating cycles



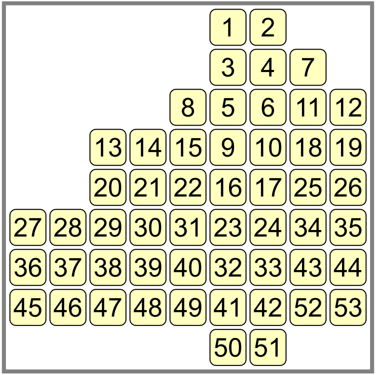
The sequence of moves R^{k}RULDL^k, with $k \geq 0$, generates a family of cycles spanning all the full rows.

Gravity puzzle: Generating cycles



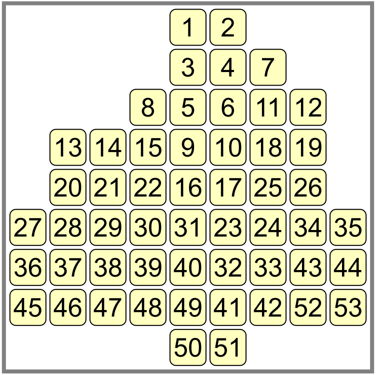
The sequence of moves R^{k}RULDL^k, with $k \geq 0$, generates a family of cycles spanning all the full rows.

Gravity puzzle: Generating cycles



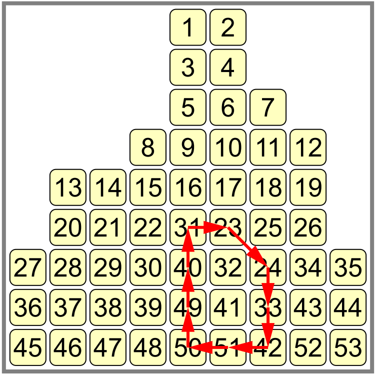
The sequence of moves $\mathsf{R}^k\mathsf{RULDL}^k$, with $k \geq 0$, generates a family of cycles spanning all the full rows.

Gravity puzzle: Generating cycles



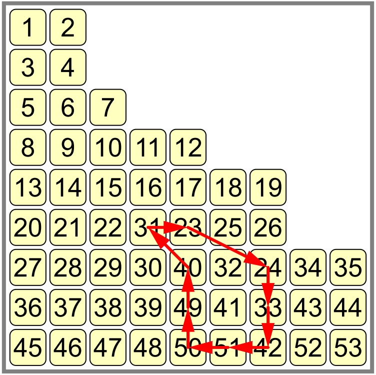
The sequence of moves $\mathsf{R}^k\mathsf{RULDL}^k$, with $k \geq 0$, generates a family of cycles spanning all the full rows.

Gravity puzzle: Generating cycles



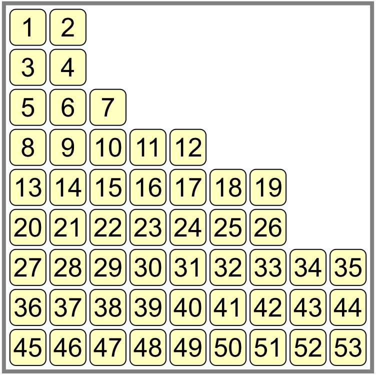
The sequence of moves $\mathrm{R}^k\mathrm{RULDL}^k$, with $k \geq 0$, generates a family of cycles spanning all the full rows.

Gravity puzzle: Generating cycles



The sequence of moves $\mathrm{R}^k\mathrm{RULDL}^k$, with $k \geq 0$, generates a family of cycles spanning all the full rows.

Gravity puzzle: Generating cycles



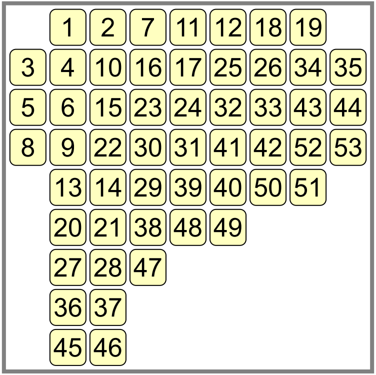
The sequence of moves $\mathsf{U}^k\mathsf{URDLD}^k$, with $k \geq 0$, generates a family of cycles spanning all the full columns.

Gravity puzzle: Generating cycles

1	2	7	11	12	18	19		
3	4	10	16	17	25	26	34	35
5	6	15	23	24	32	33	43	44
8	9	22	30	31	41	42	52	53
13	14	29	39	40	50	51		
20	21	38	48	49				
27	28	47						
36	37							
45	46							

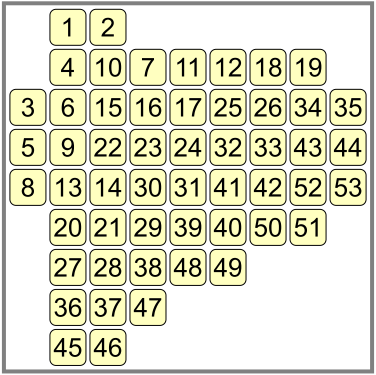
The sequence of moves U^kURDLD^k, with $k \geq 0$, generates a family of cycles spanning all the full columns.

Gravity puzzle: Generating cycles



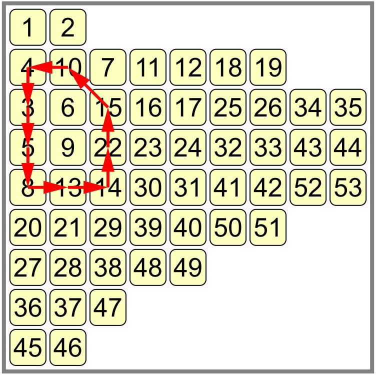
The sequence of moves U^kURDLD^k, with $k \geq 0$, generates a family of cycles spanning all the full columns.

Gravity puzzle: Generating cycles



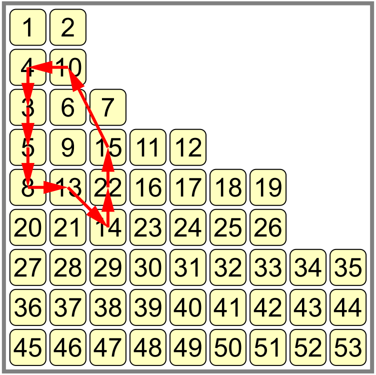
The sequence of moves $\mathsf{U}^k\mathsf{URDLD}^k$, with $k \geq 0$, generates a family of cycles spanning all the full columns.

Gravity puzzle: Generating cycles



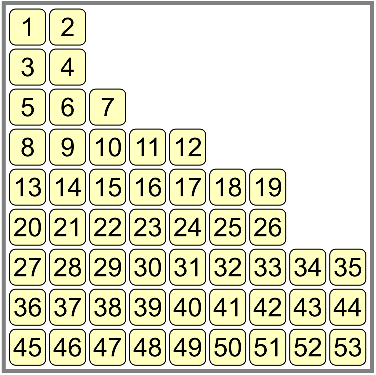
The sequence of moves $\mathsf{U}^k\mathsf{URDLD}^k$, with $k \geq 0$, generates a family of cycles spanning all the full columns.

Gravity puzzle: Generating cycles



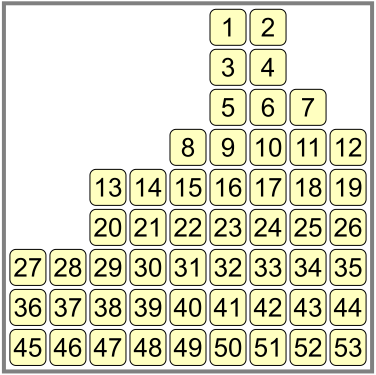
The sequence of moves $\mathsf{U}^k\mathsf{URDLD}^k$, with $k \geq 0$, generates a family of cycles spanning all the full columns.

Gravity puzzle: Generating cycles



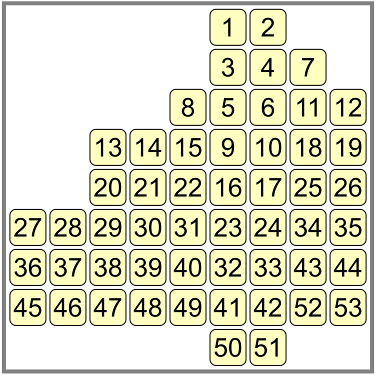
The sequence of moves $\mathsf{R}^k\mathsf{URDLL}^k$, with $k \geq 0$, generates a family of cycles spanning all the non-full rows and columns.

Gravity puzzle: Generating cycles



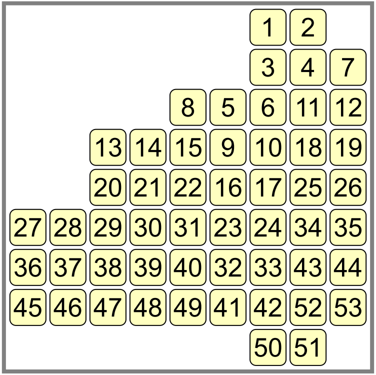
The sequence of moves $\mathsf{R}^k\mathsf{URDLL}^k$, with $k \geq 0$, generates a family of cycles spanning all the non-full rows and columns.

Gravity puzzle: Generating cycles



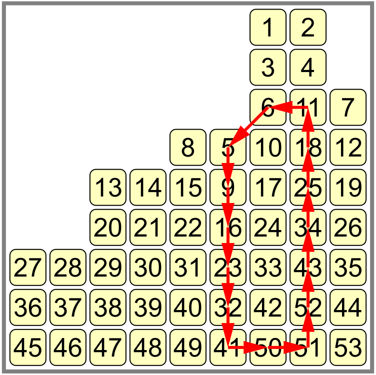
The sequence of moves R^kURDLL^k, with $k \geq 0$, generates a family of cycles spanning all the non-full rows and columns.

Gravity puzzle: Generating cycles



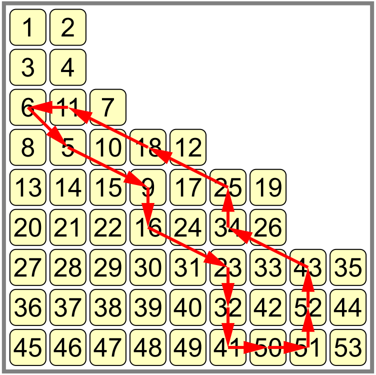
The sequence of moves R^{k}URDLL^k, with $k \geq 0$, generates a family of cycles spanning all the non-full rows and columns.

Gravity puzzle: Generating cycles



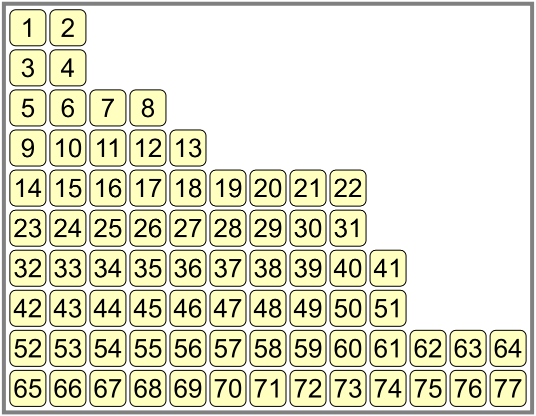
The sequence of moves $\mathsf{R}^k\mathsf{URDLL}^k$, with $k \geq 0$, generates a family of cycles spanning all the non-full rows and columns.

Gravity puzzle: Generating cycles



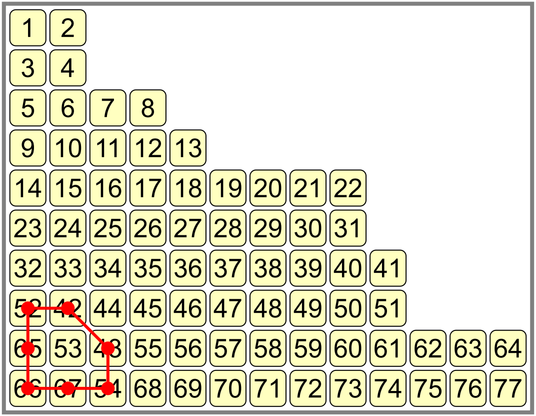
The sequence of moves $\mathsf{R}^k\mathsf{URDLL}^k$, with $k \geq 0$, generates a family of cycles spanning all the non-full rows and columns.

Gravity puzzle: Generating cycles



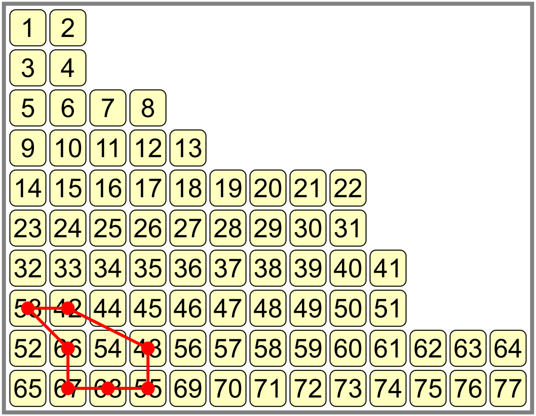
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



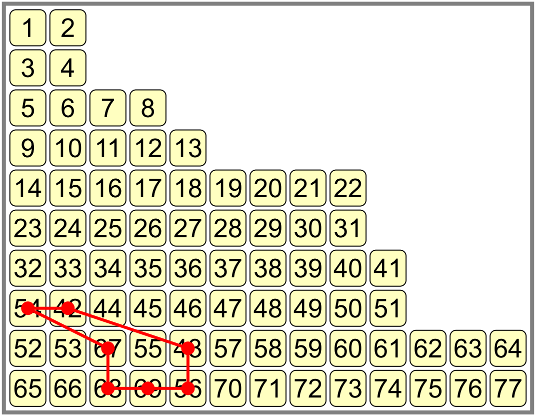
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



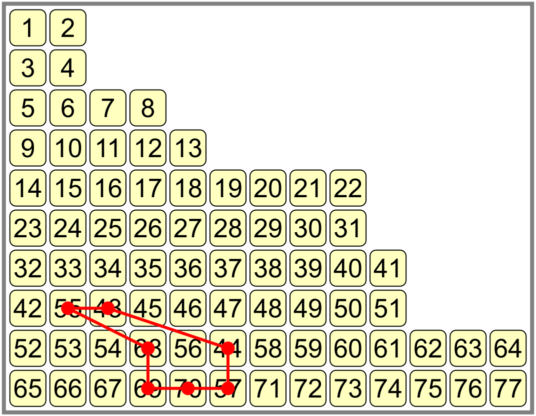
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



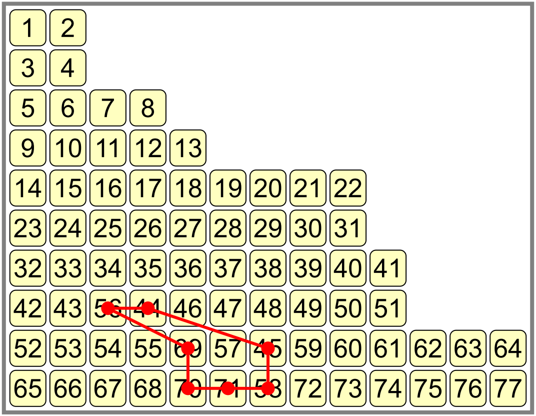
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles

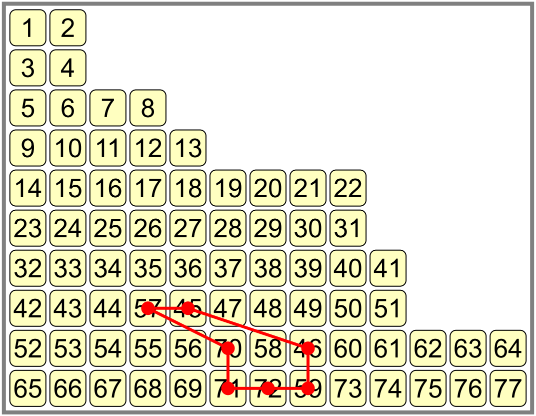


These cycles collectively span all tokens.

Gravity puzzle: Generating cycles

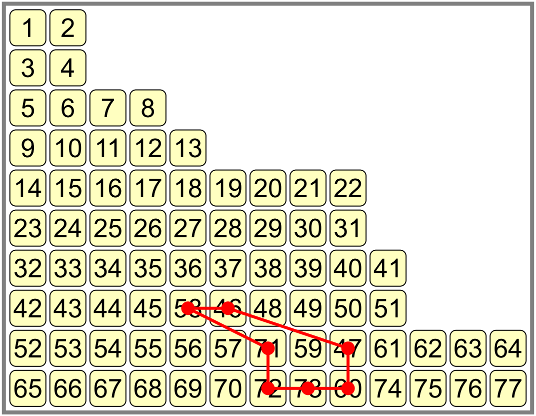


These cycles collectively span all tokens.



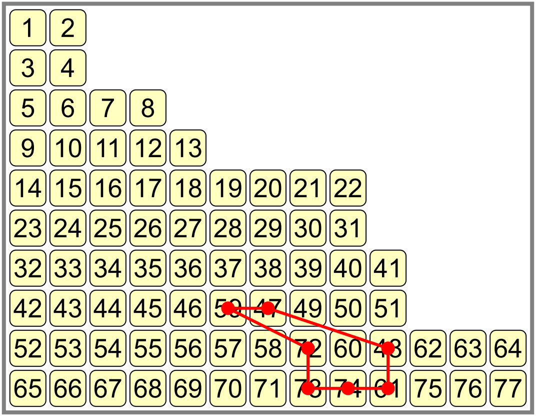
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



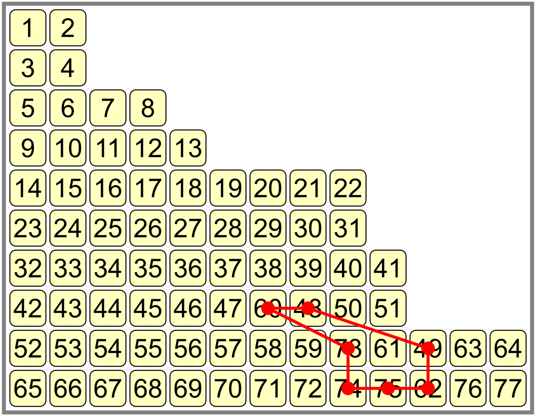
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



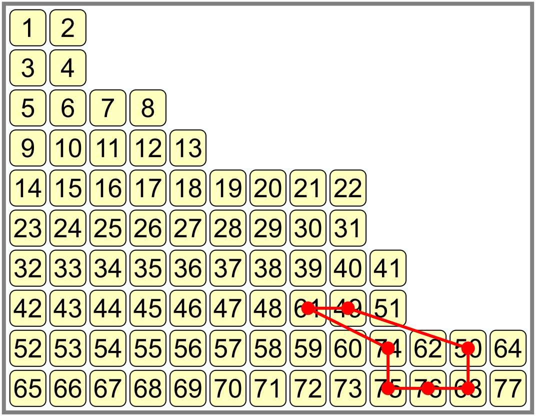
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



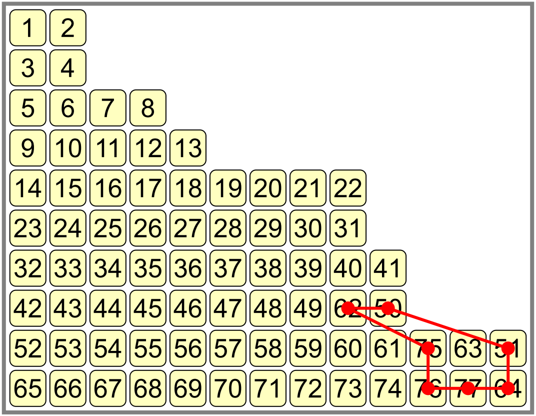
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



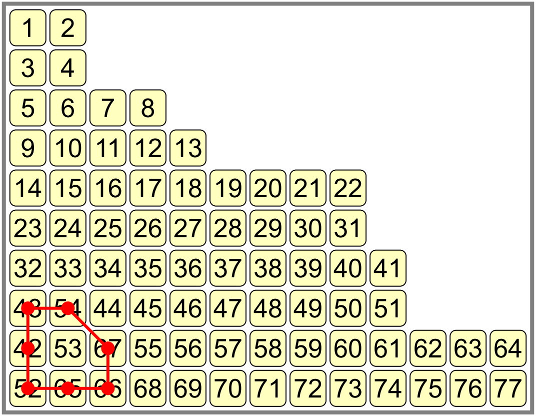
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



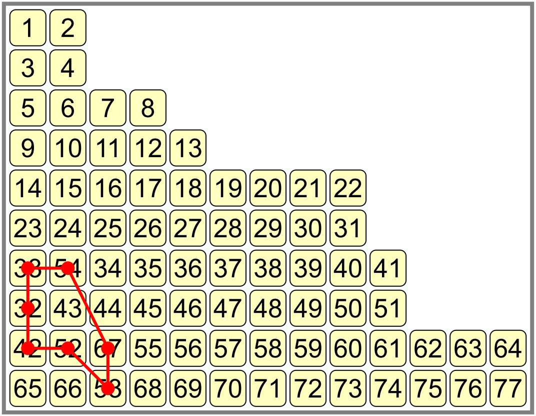
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



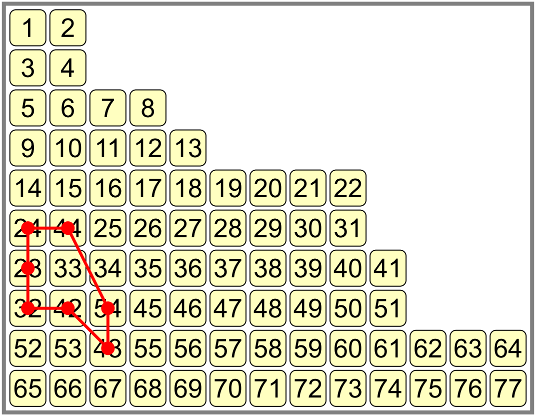
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



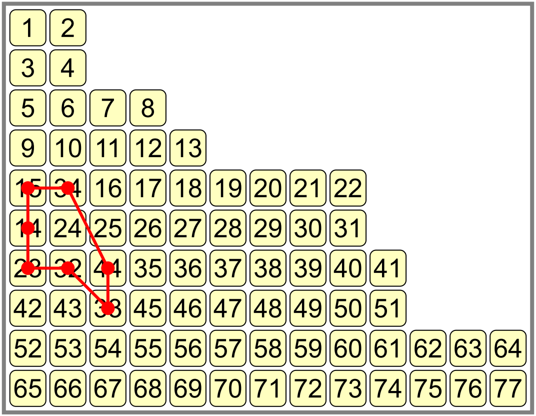
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles

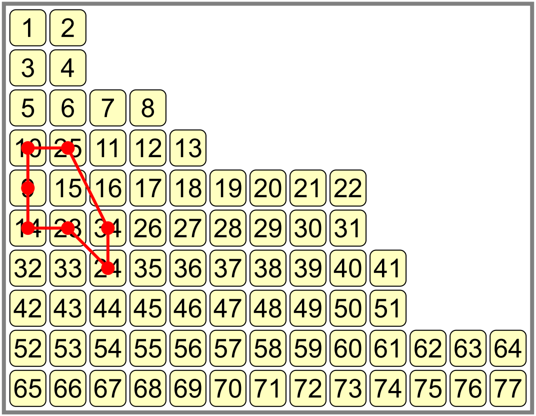


These cycles collectively span all tokens.

Gravity puzzle: Generating cycles

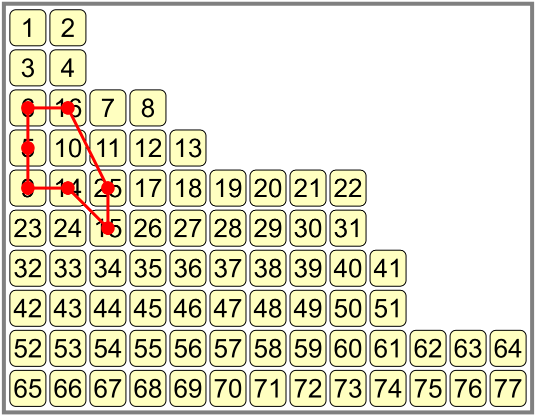


These cycles collectively span all tokens.



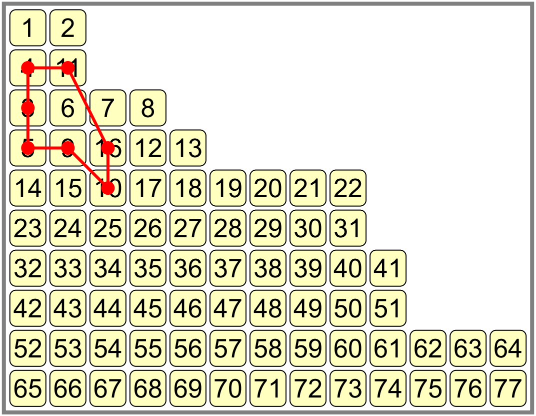
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



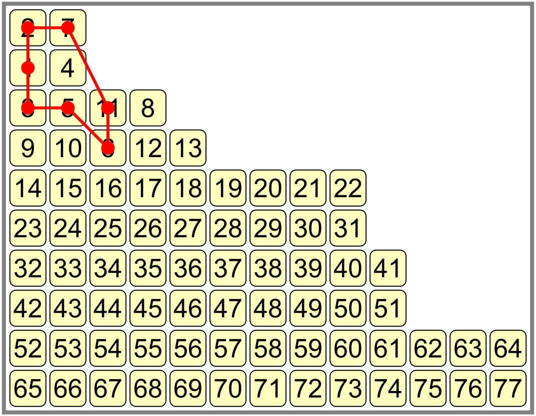
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



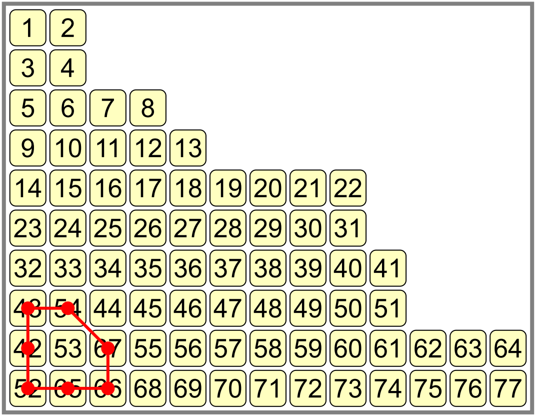
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



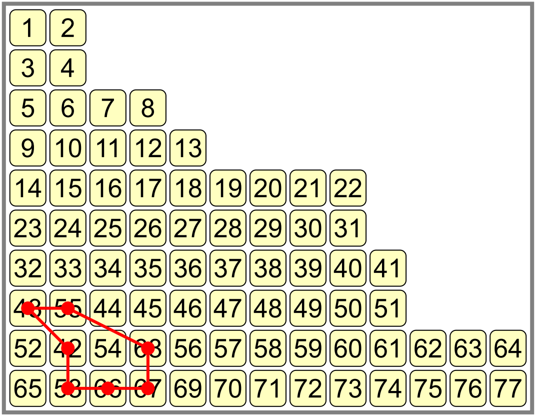
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



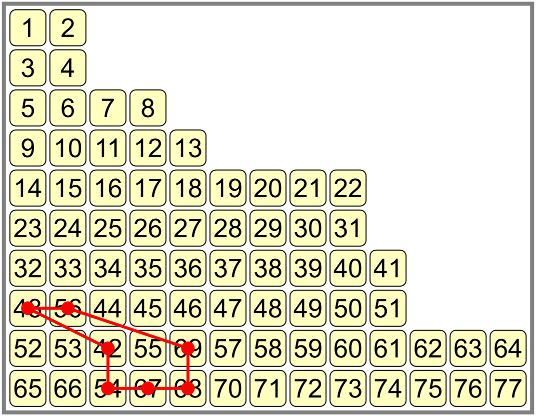
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



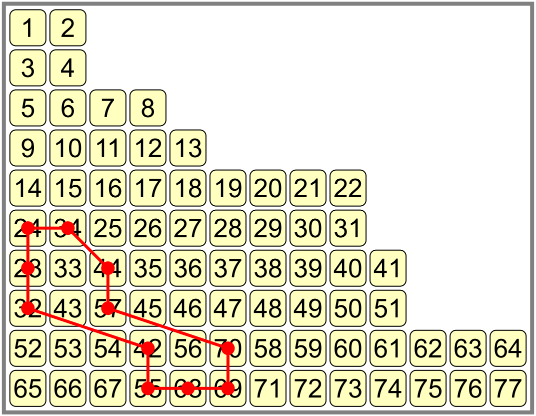
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



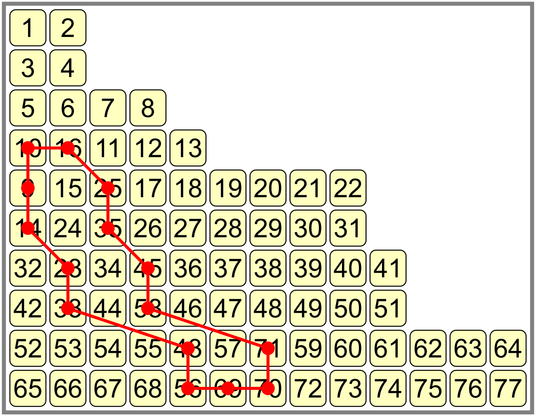
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



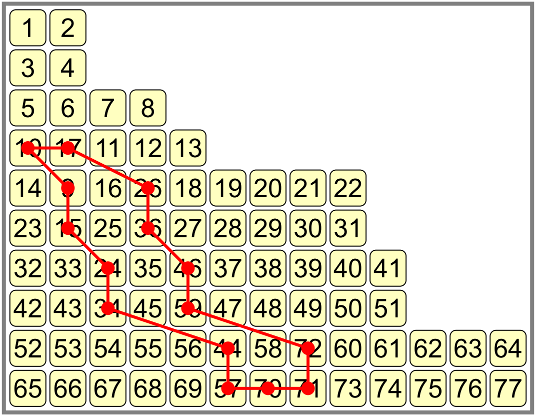
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



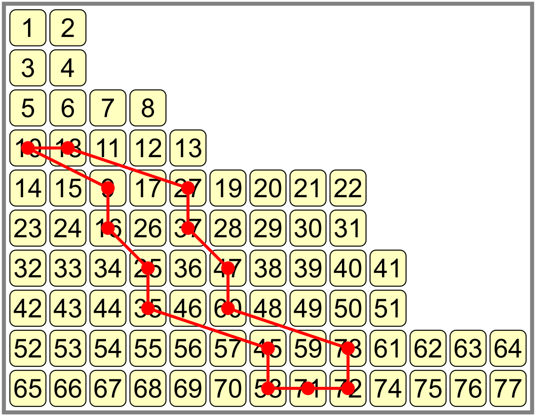
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles

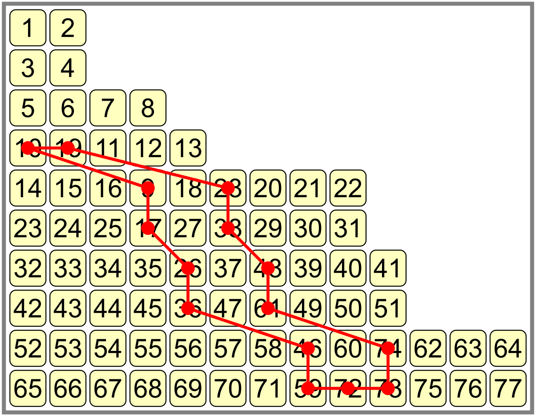


These cycles collectively span all tokens.

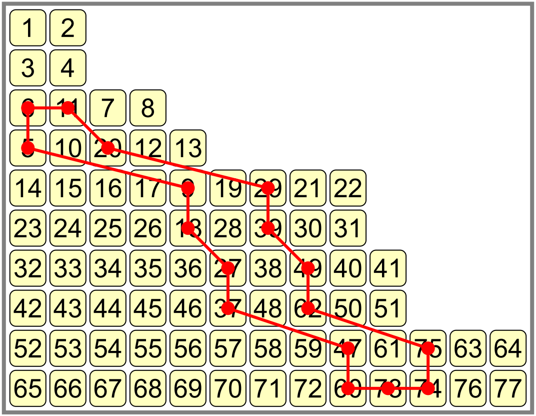
Gravity puzzle: Generating cycles



These cycles collectively span all tokens.

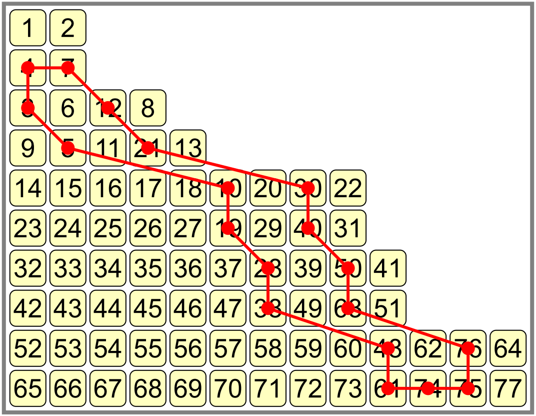


These cycles collectively span all tokens.



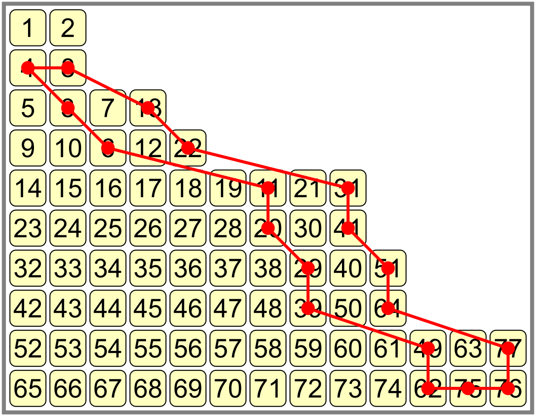
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



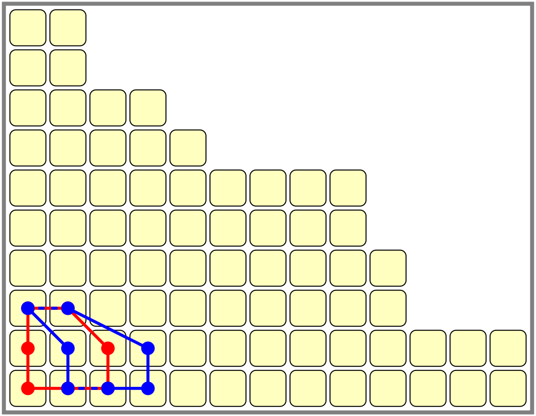
These cycles collectively span all tokens.

Gravity puzzle: Generating cycles



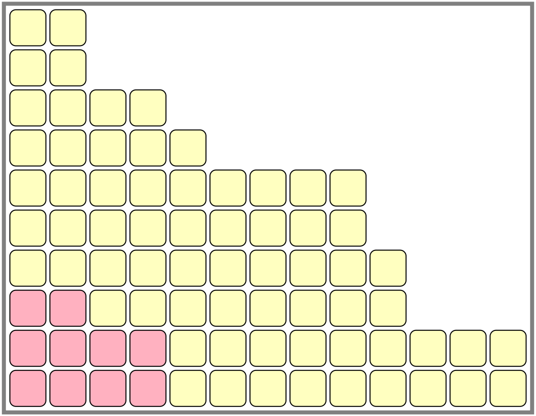
These cycles collectively span all tokens.

Gravity puzzle: Generating all even permutations



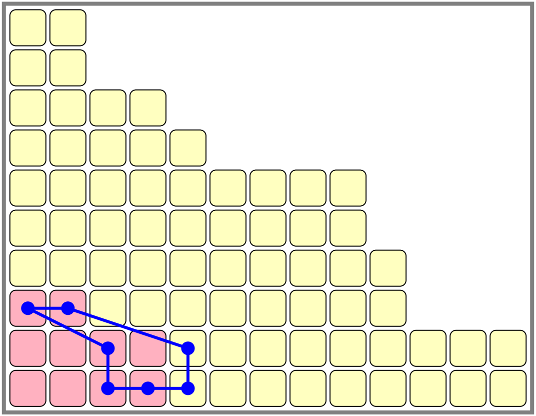
By our first lemma, these two cycles determine a 2-transitive permutation group acting on the tokens they span.

Gravity puzzle: Generating all even permutations



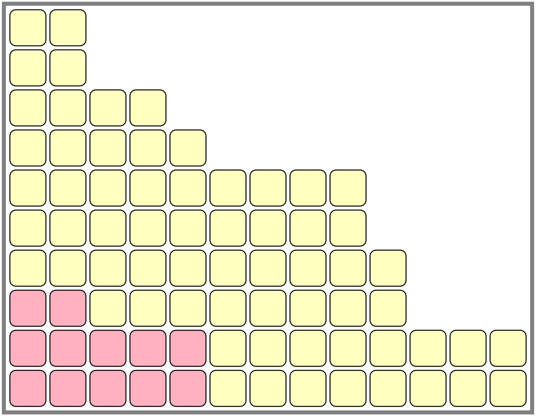
By our first lemma, these two cycles determine a 2-transitive permutation group acting on the tokens they span.

Gravity puzzle: Generating all even permutations



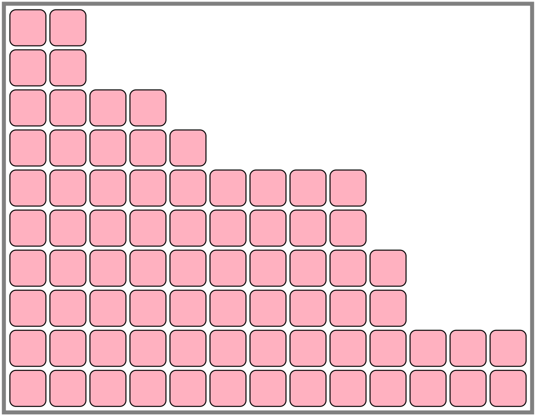
By our second lemma, we can extend this set to the tokens spanned by the next cycle.

Gravity puzzle: Generating all even permutations



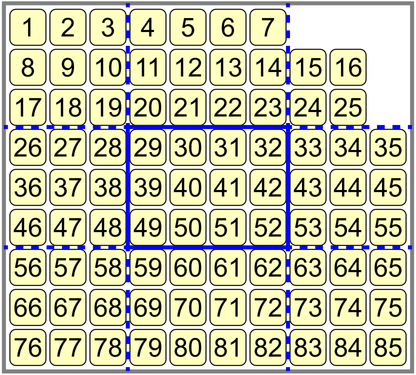
Proceeding in this fashion, we conclude that our cycles act 2-transitively on all the tokens they span.

Gravity puzzle: Generating all even permutations



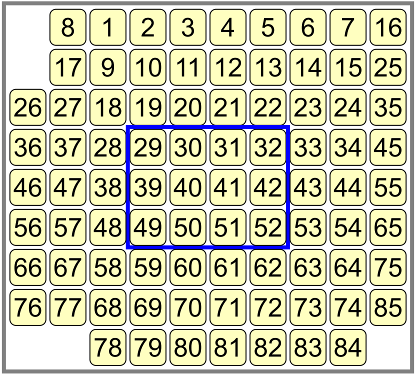
Thus, by Jones' theorem, we can generate all even permutations on the tokens spanned by our cycles.

Gravity puzzle: Unmovable core

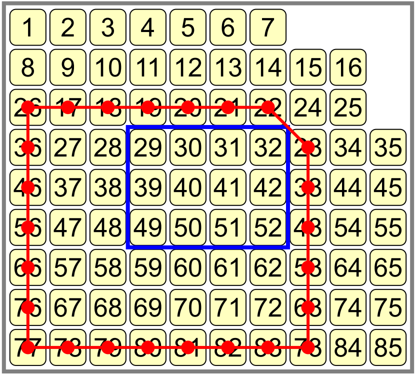


In general, if more than half of the rows and more than half of the columns are full, there is a *central core* that is impossible to move.

Gravity puzzle: Unmovable core

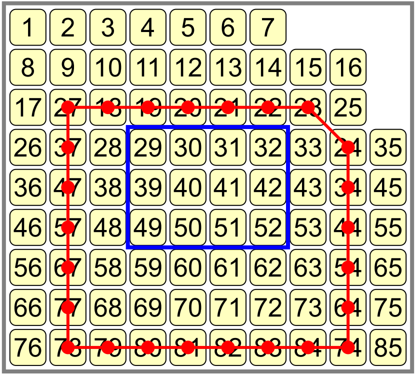


In general, if more than half of the rows and more than half of the columns are full, there is a *central core* that is impossible to move.



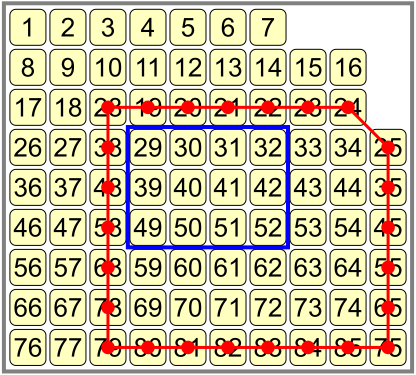
However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

Gravity puzzle: Unmovable core

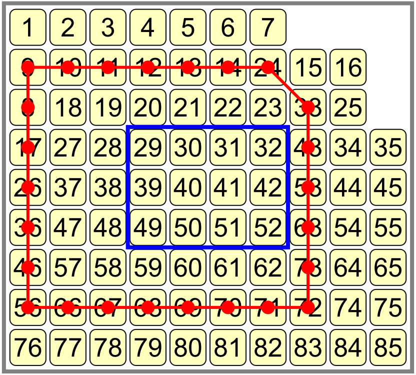


However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

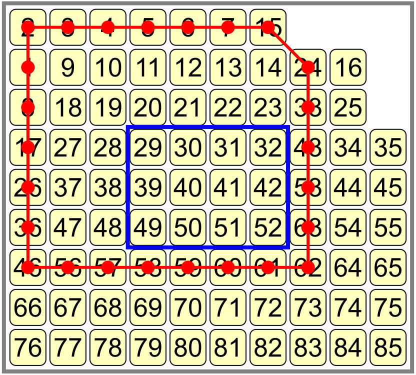
Gravity puzzle: Unmovable core



However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

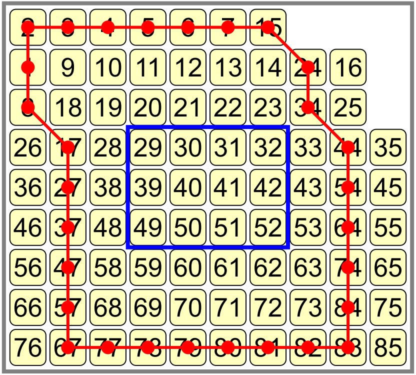


However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.



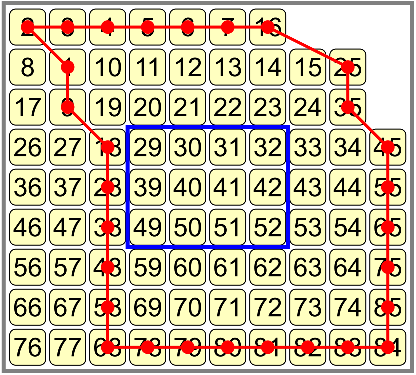
However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

Gravity puzzle: Unmovable core



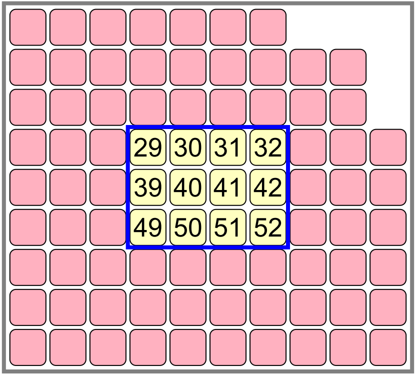
However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

Gravity puzzle: Unmovable core



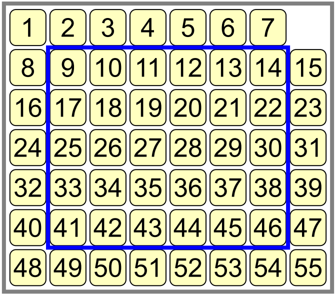
However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

Gravity puzzle: Unmovable core



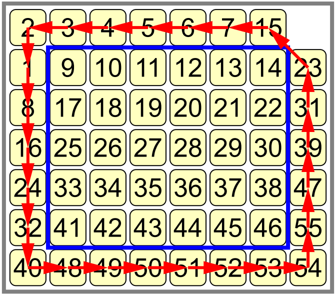
However, there are cycles spanning all non-core tokens. Thus, we can generate all even permutations of the non-core tokens.

Gravity puzzle: Special cases



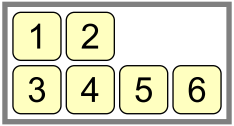
An exception is when there is exactly one empty cell. In this case, the permutation group is easily seen to be cyclic.

Gravity puzzle: Special cases



An exception is when there is exactly one empty cell. In this case, the permutation group is easily seen to be cyclic.

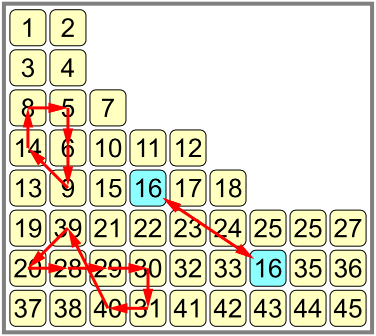
Gravity puzzle: Special cases



The only other exception is the puzzle with a 4×2 bounding box and two empty cells. In this case, the permutation group has order 60 and is isomorphic to A_5.

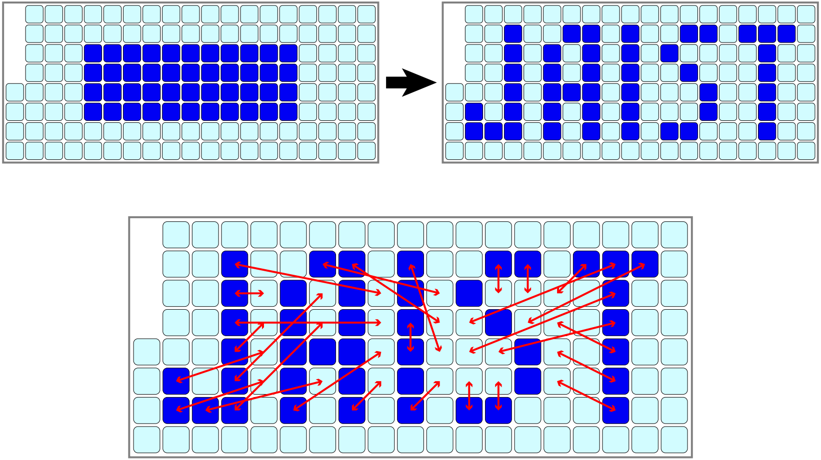
Gravity puzzle: Non-unique labels

What if not all labels are unique? Note that we can generate a transposition between two equal (non-core) tokens "for free". The resulting group *strictly* contains all even permutations, and thus contains *all* permutations of the (non-core) tokens!*



*Recall that A_n has index 2 in S_n. By Lagrange's theorem, any subgroup of S_n strictly containing A_n must have index 1, and therefore it must be S_n itself.

Gravity puzzle: Non-unique labels



In this puzzle, since the core is empty and some tokens are equal, *all* permutations of the tokens are possible.

Conclusion and open problems

Theorem (Sparse configurations)

All sparse configurations of $n = ab$ tokens can be pushed into an $a \times b$ box if and only if $a \leq 2$ or $b \leq 2$ or $a = b = 3$.

Open problem

Is it NP-complete to decide whether a given sparse configuration can be pushed into a given rectangle?

Theorem (Reconfiguration)

If the configuration is compact, then the possible permutations are:

- *If exactly one cell is empty, the cycles of the non-core tokens.*
- *If there are 6 tokens and 2 empty cells, a group isomorphic to A_5.*
- *Otherwise, if all non-core tokens are distinct, all the even permutations of the non-core tokens.*
- *Otherwise, all permutations of the non-core tokens.*

Open problem

What if the tokens form a non-compact configuration?