Simply Connected Monohedral Polyhedra

Tutorial on Discrete Geometry and Graph Theory

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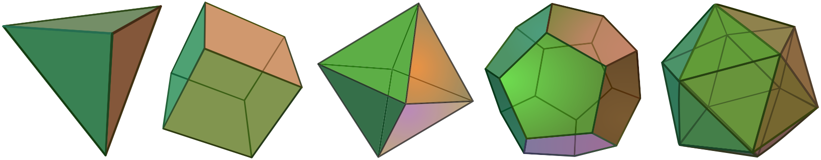
Monohedral Polyhedra

A polyhedron whose faces are all congruent is said *monohedral*.

Open Problem (CCCG 2022)

Do all monohedral polyhedra have an even number of faces?

Example: The Platonic solids have 4, 6, 8, 12, 20 faces.



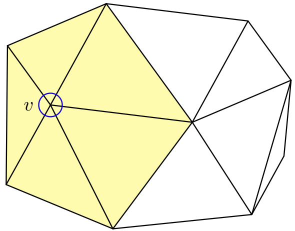
Theorem (Grünbaum, 1960)

*All **convex** monohedral polyhedra have an even number of faces.*

Definition of Polyhedron

A *polyhedron* is a bounded solid figure satisfying these conditions:

- Its boundary consists of finitely many polygons called *faces*.
- The intersection of two faces is either empty or a common *edge* or a common *vertex* (faces are internally disjoint).
- Each edge is shared by exactly two faces.
- The faces meeting at each vertex form a single *circuit*.

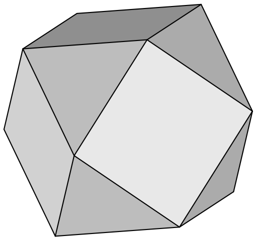


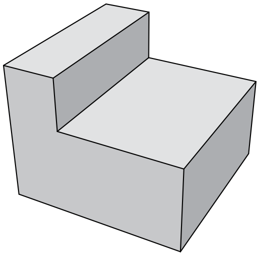
The last two conditions can be equivalently summarized as:

- The surface of a polyhedron is a *2-manifold*.

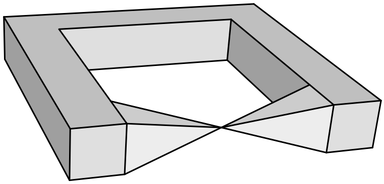
Definition of Polyhedron

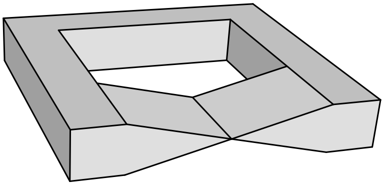
Examples: A *convex* polyhedron and a *non-convex* polyhedron:





The following are *not* considered polyhedra:

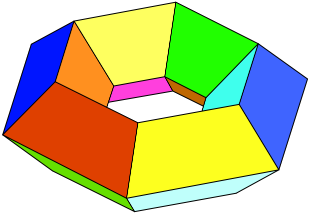


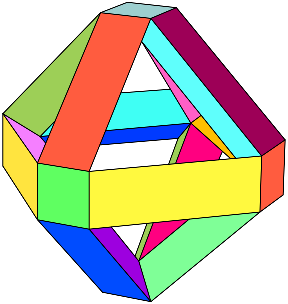


Simply Connected Polyhedra

A polyhedron is *simply connected* if its surface can be continuously deformed to become a sphere.

- All convex polyhedra are simply connected.
- Simply connected polyhedra have no *holes* or *handles*.
- The following polyhedra are *not* simply connected:

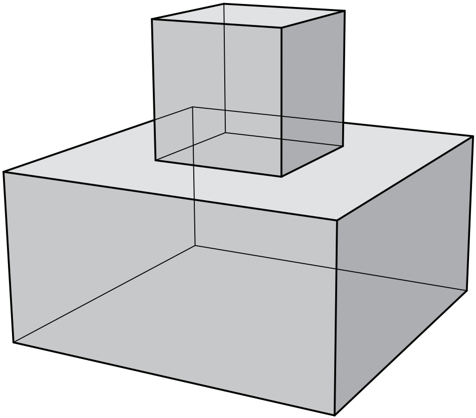




In this seminar, we will focus on simply connected polyhedra.

Faces with Holes

Even if a polyhedron is simply connected, its faces may have *holes*:

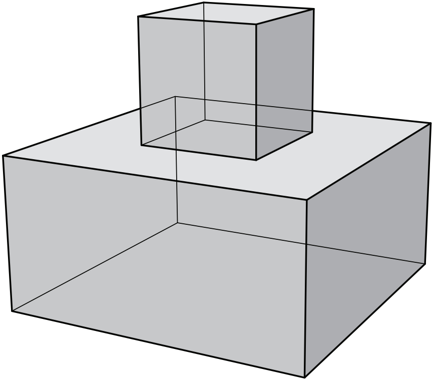


Euler's Formula

Theorem (Euler, 1758)

For any simply connected polyhedron with f faces, e edges, v vertices, and a total of h face holes, we have $f + v - e - h = 2$.

Proof.



Triangulate all faces by adding new edges between existing vertices.

Euler's Formula

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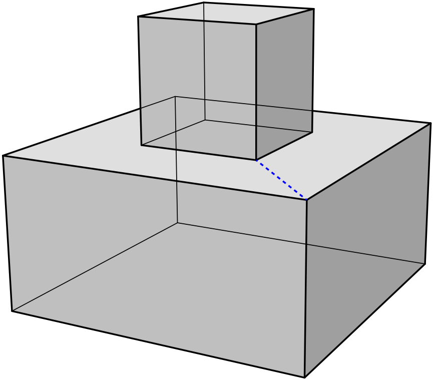

If a new edge eliminates a hole, we have $e \leftarrow e + 1$ and $h \leftarrow h - 1$.

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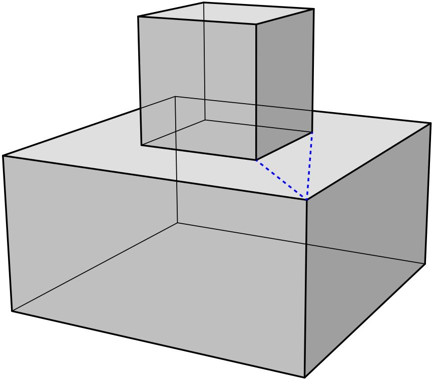

Otherwise, we have $e \leftarrow e + 1$ and $f \leftarrow f + 1$.

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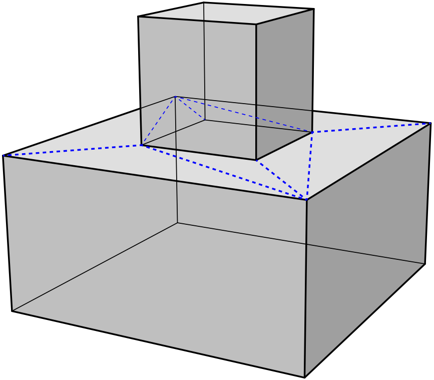

Either way, the quantity $f + v - e - h$ is preserved.

Euler's Formula

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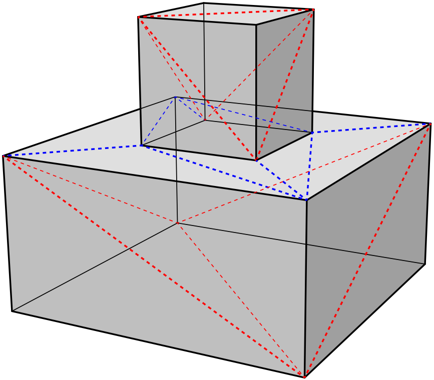
Proof.



Thus, we may assume that all faces are triangles, and hence $h = 0$.

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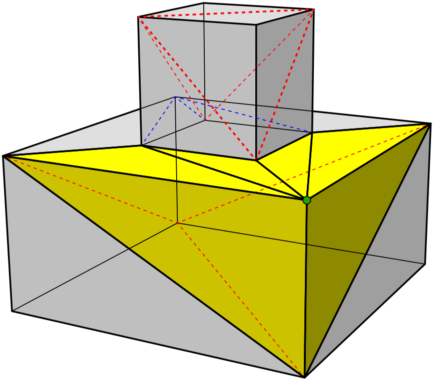

We will prove the formula by induction on v. Choose any vertex.

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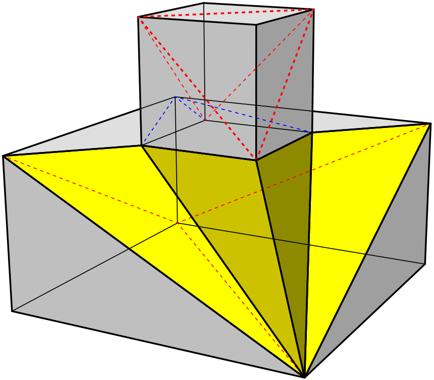

Remove its incident faces and add new triangles to cover the hole.

Euler's Formula

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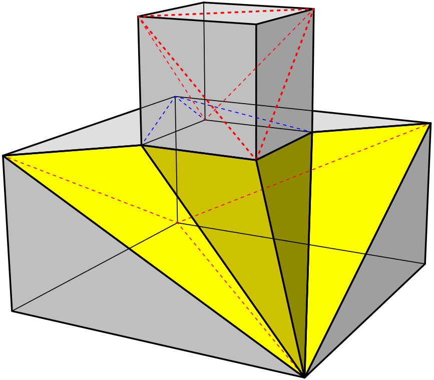

As a result, $v \leftarrow v - 1$, $f \leftarrow f - 2$, and $e \leftarrow e - 3$.

Euler's Formula

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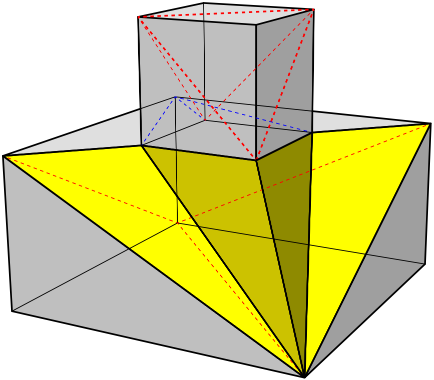

Again, $f + v - e$ is preserved, and by induction $f + v - e = 2$.

Euler's Formula

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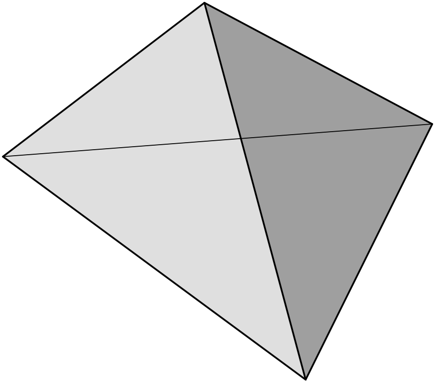
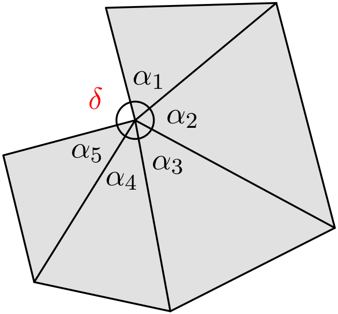


The base case is a tetrahedron, where $f = 4$, $v = 4$, and $e = 6$. □

Angular Defect

If the face angles incident to a vertex v of a polyhedron are α_1, α_2, $\dots$, α_k, we define the *angular defect* of v as

$$\delta_v = 2\pi - \sum_{i=1}^{k} \alpha_i$$



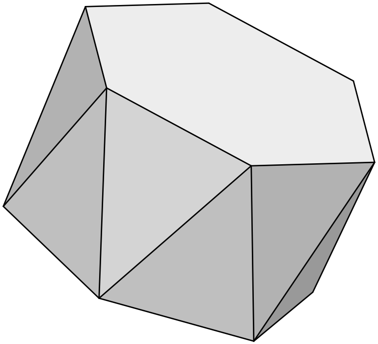
Note that the angular defect is at most 2π and may be negative.

Descartes' Theorem (i.e., Polyhedral Gauss–Bonnet)

Theorem (Descartes, ∼1630)

For any simply connected polyhedron, we have $\sum \delta_v = 4\pi$*, where the sum ranges through all vertices of the polyhedron.*

Proof.



Triangulate all faces of the polyhedron. Note that $\sum \delta_v$ remains unchanged.

Descartes' Theorem (i.e., Polyhedral Gauss–Bonnet)

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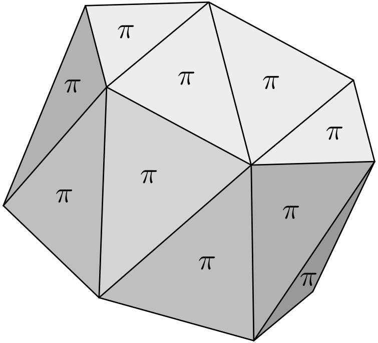

Recall that the sum of angles of a triangle is π. Thus, the sum of all face angles is $\sum \alpha_i = \pi f$.

Descartes' Theorem (i.e., Polyhedral Gauss–Bonnet)

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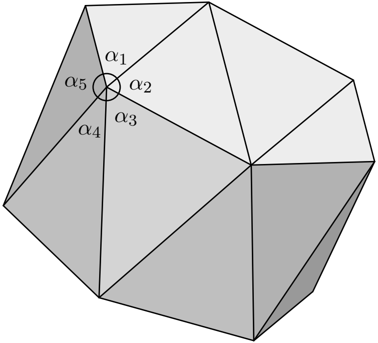


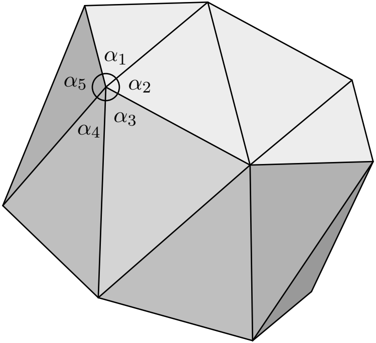
By regrouping the terms of the sum, we have
$\sum \delta_v = 2\pi v - \sum \alpha_i = 2\pi v - \pi f$.

Descartes' Theorem (i.e., Polyhedral Gauss–Bonnet)

Theorem (Descartes, ∼1630)

For any simply connected polyhedron, we have $\sum \delta_v = 4\pi$, *where the sum ranges through all vertices of the polyhedron.*

Proof.



Since each face has 3 edges and each edge is shared by 2 faces, we have $3f = 2e$. Euler's formula becomes $f + v - e = v - f/2 = 2$.

Descartes' Theorem (i.e., Polyhedral Gauss–Bonnet)

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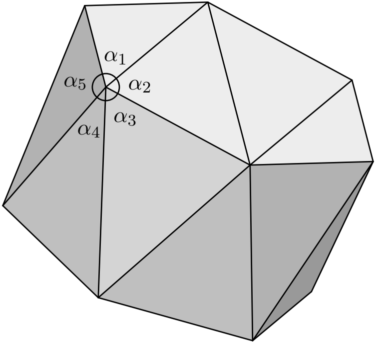
Proof.



Therefore, $\sum \delta_v = 2\pi v - \pi f = 2\pi(v - f/2) = 4\pi$. □

Roadmap

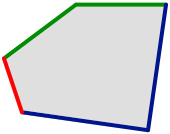
Let us assume that $\mathcal{P}$ is a simply connected monohedral polyhedron with an *odd* number of faces. Let F be a face of $\mathcal{P}$.

We will prove the following:

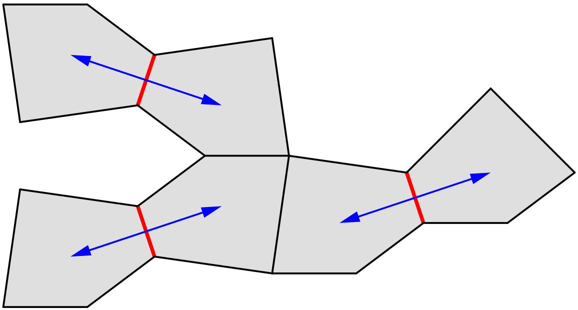
- F cannot have a distinguished edge.
- F cannot have holes.
- F cannot have an odd number of edges.
- F must have 4 edges.
- F must be a parallelogram, a dart, or a kite.
    - F cannot be a parallelogram.
    - F cannot be a dart.
    - Can F be a kite?

Faces with a Distinguished Edge

Assume that a face of the polyhedron has an edge whose length is different from all others.



Then, all faces can be paired up, and therefore they are even.



Bounding the Number of Face Edges

If each face of the polyhedron has at least one hole, then $h \geq f$.
Each vertex is incident to at least 2 edges, and each edge is incident to exactly 2 vertices. Thus, $2v \leq 2e$.
By Euler's formula, $2 = f + v - e - h \leq v - e \leq 0$, a contradiction.
We conclude that **faces have no holes**, i.e., $h = 0$.

Let k be the number of edges of each face of the polyhedron.
We have $kf = 2e$, because each edge is shared by exactly 2 faces.
Therefore, if the number of faces is odd, then k **must be even**.

Assume that all vertices of the polyhedron have at least 3 incident edges. Thus, $3v \leq 2e$, because each edge is incident to 2 vertices.
By Euler's formula, $2 = f + v - e$ (recall that $h = 0$).

$6 = 3f + 3v - 3e \leq 3f + 2e - 3e = 3f - e$

$12 \leq 6f - 2e = 6f - kf$

$k \leq 6 - 12/f < 6$.

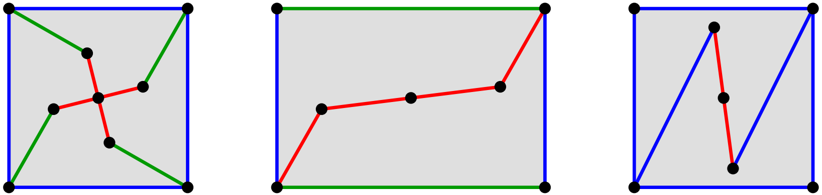
We conclude that $k < 6$, and hence **faces must be quadrilaterals**.

Vertices of Degree 2

If the polyhedron has some vertices of degree 2, we can eliminate them and merge co-planar faces.

The resulting polyhedron has no vertices of degree 2.

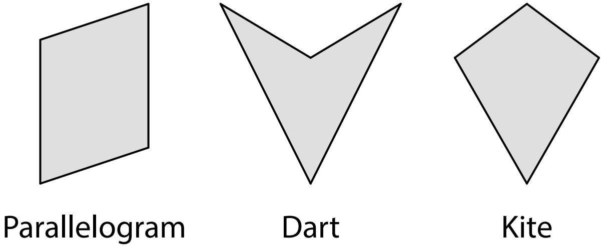
By the previous argument, it has a face with fewer than 6 edges.



A case analysis allows us to conclude that, also in this situation, **faces must be quadrilaterals** *(details omitted)*.

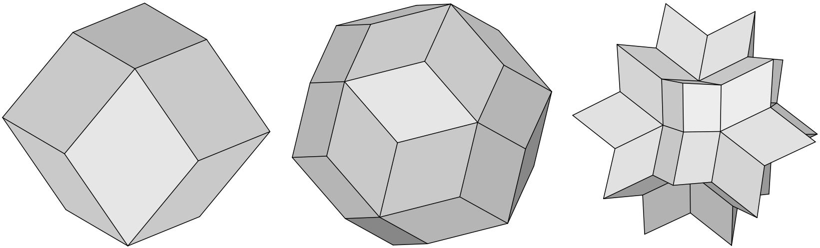
Allowed Faces: Parallelogram, Dart, and Kite

Summarizing, the faces must be of one of these three types:



Parallelogram Faces

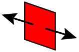
Assume that all faces are parallelograms.



We will prove that the number of faces must be even.

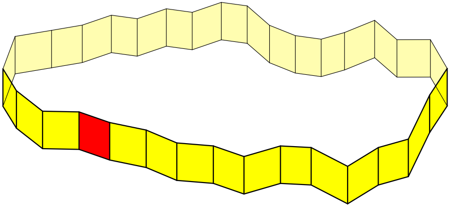
(This is also true if the parallelograms are not all congruent.)

Parallelogram Faces



Each face determines a *zone* of faces with parallel opposite edges that divides the polyhedron into an “upper” and a “lower” region.

Parallelogram Faces



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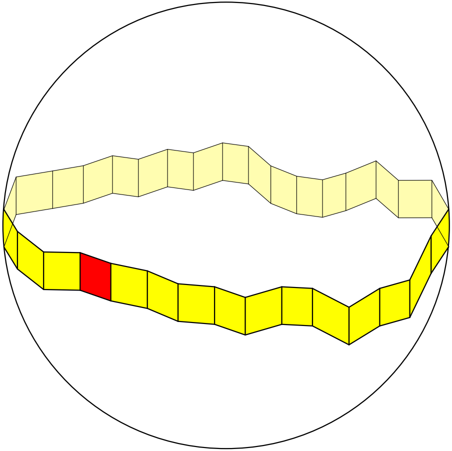

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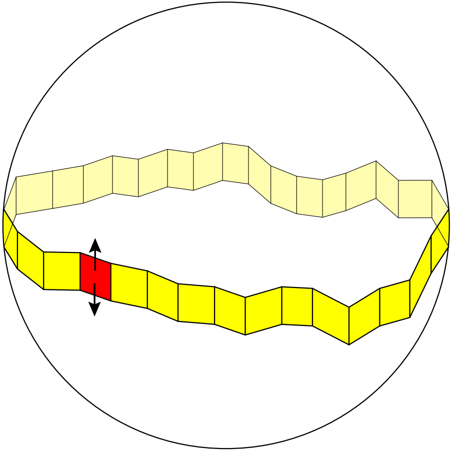
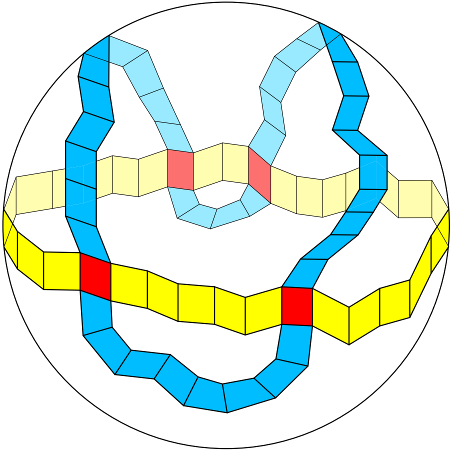


This face also determines a second zone that crosses the first zone an even number of times (moving between upper and lower region).

Parallelogram Faces



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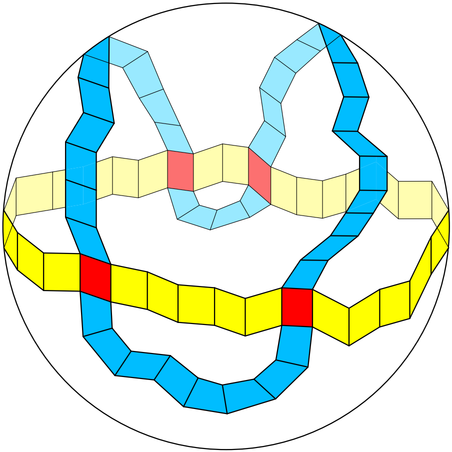
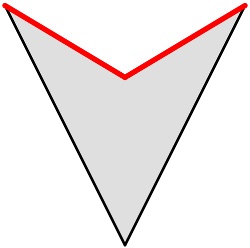


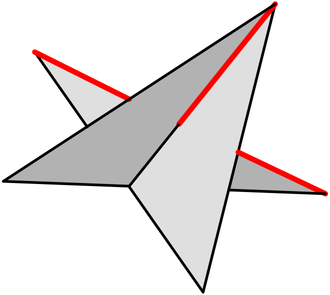
Since each face belongs to exactly two zones, we can partition the set of faces into subsets of even size. Thus, **the faces are even**.

Dart Faces



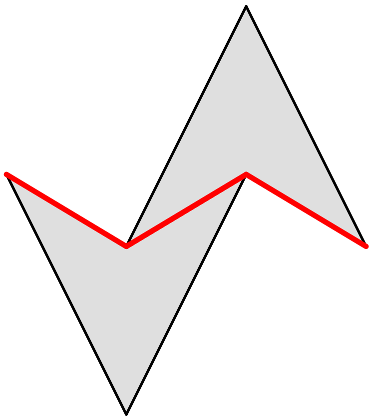
Assume that all faces are darts. Let us focus on the *short edges*.

Dart Faces



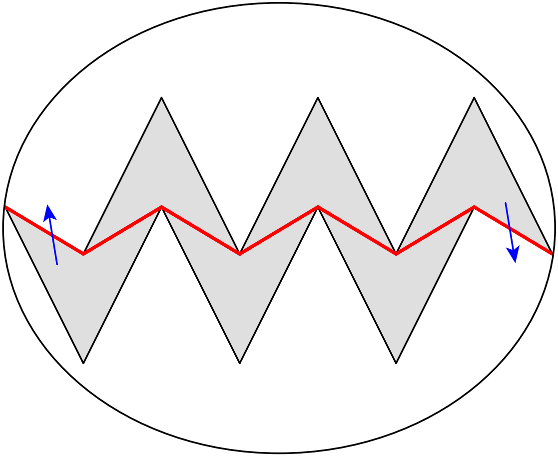
Two darts sharing a short edge cannot be “concordant”, or they would not be internally disjoint.

Dart Faces



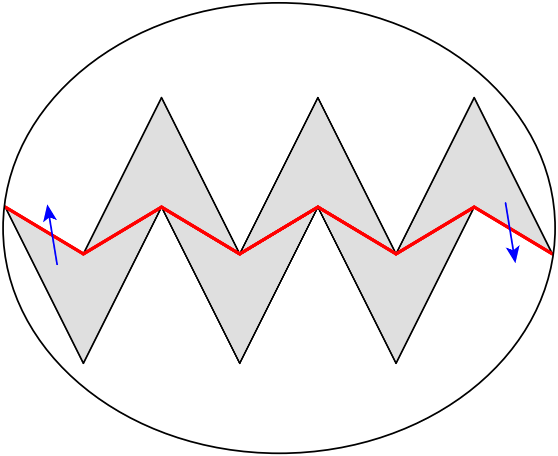
So, darts sharing short edges should be "discordant" and form closed *chains* of faces.

Dart Faces



The line of short edges in a chain divides the polyhedron in two parts, with darts alternately pointing to one side or the other.

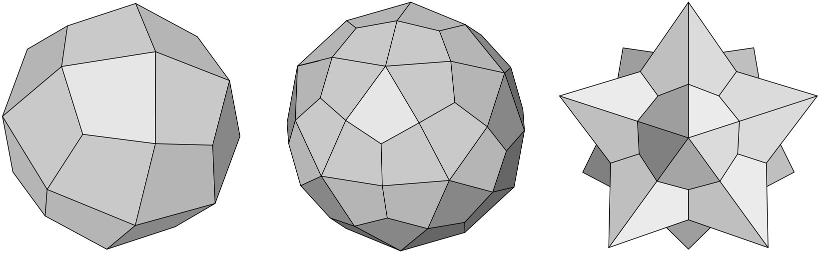
Dart Faces



Thus, we can partition the set of faces into chains, and each chain has even size. We conclude that **the faces are even**.

Kite Faces

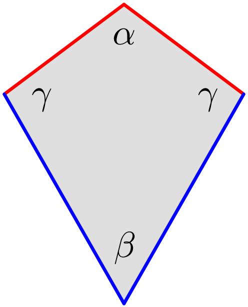
Assume that all faces are kites.



We will analyze the combinatorial structure of these polyhedra.

Kite Faces

We may assume that the kite's edges are not all equal, or it would be a parallelogram. Hence, it has two short and two long edges.



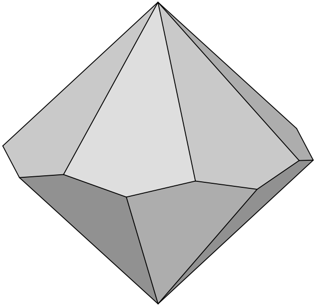
Kite Faces: Apexes

An *apex* is a vertex whose incident edges have all the same length.

Theorem

Any simply connected monohedral polyhedron whose faces are kites has at least two apexes.

Proof.



Draw a *node* on each face, and connect nodes on adjacent faces. This is called the *dual graph* of the polyhedron.

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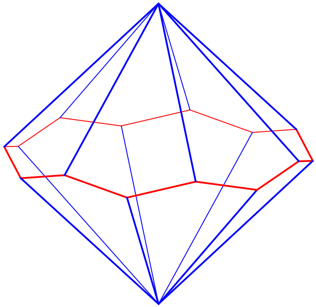

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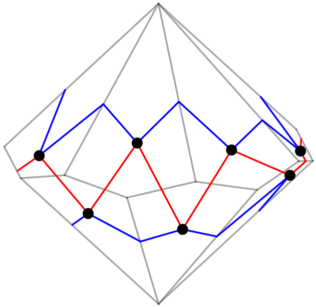

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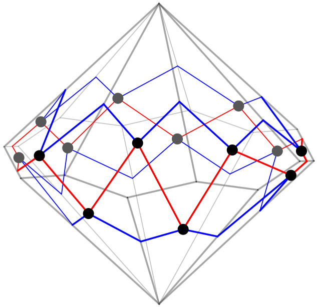
Proof.



Draw a *node* on each face, and connect nodes on adjacent faces. This is called the *dual graph* of the polyhedron.

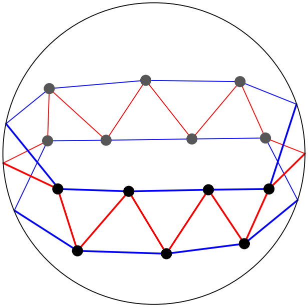
Kite Faces: Apexes

An *apex* is a vertex whose incident edges have all the same length.

Theorem

Any simply connected monohedral polyhedron whose faces are kites has at least two apexes.

Proof.



Observe that the edges corresponding to short (resp. long) edges induce a *vertex-disjoint cycle cover* of the dual graph.

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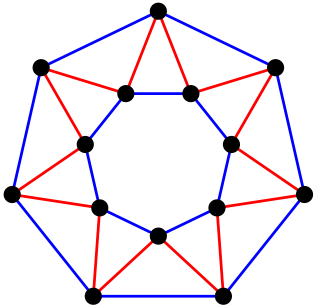
Proof.



The cycles of one cover never cross the cycles of the other cover, so all the edges of a cycle are on the same side of any other cycle.

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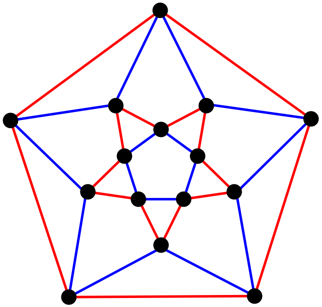
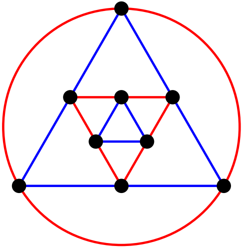
Proof.



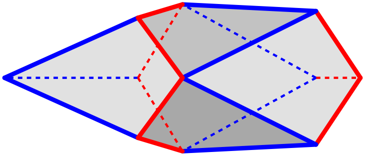
Since the dual graph is spherical, there must be at least two cycles with no cycles on one side. These cycles correspond to apexes. □

Kite Faces

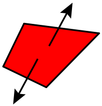
Observe that this technique is insufficient to conclude that there must be an even number of faces. This is a counterexample:



Furthermore, both Euler's formula and Descartes' theorem are insufficient to prove that this cannot be a monohedral polyhedron:

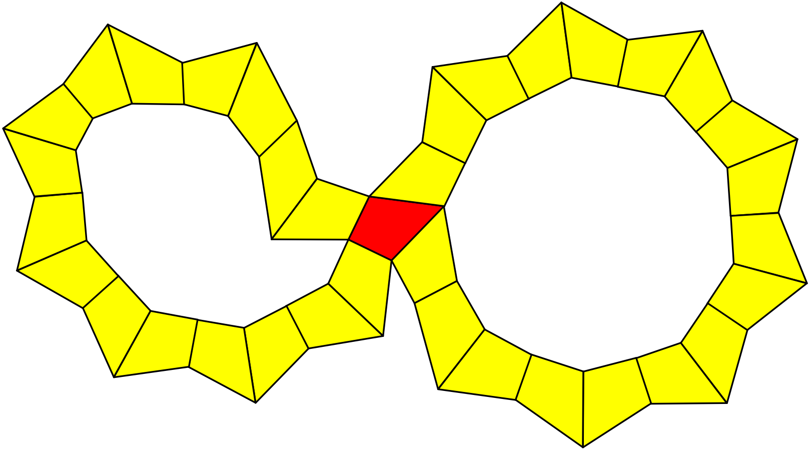


Kite Faces: Zones



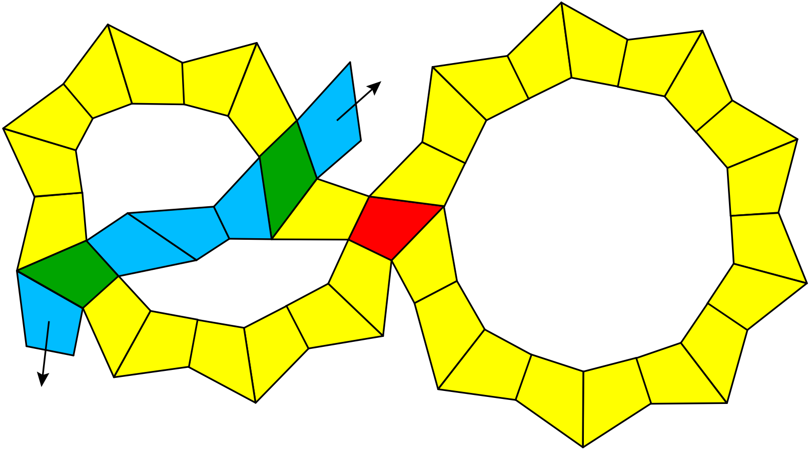
Let us define *zones* in the same way as we did for polyhedra with parallelogram faces. Here, a zone alternates short and long edges.

Kite Faces: Zones



Again, a zone is a cycle of kites. However, since kites do not have parallel opposite edges, a zone may self-intersect multiple times.

Kite Faces: Zones



How do different zones interact with each other? Do they always intersect in an even number of kites?

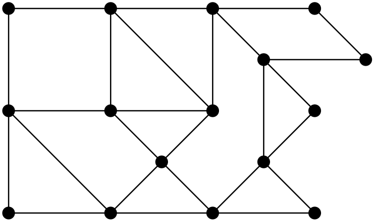
Eulerian Graphs and their Duals

A connected graph is *Eulerian* if all its vertices have even degree.
A graph is *bipartite* if all its cycles have even length.

Lemma

Any plane Eulerian graph has a bipartite dual graph.

Proof.



Note that the dual graph can be drawn as a plane graph, as well.

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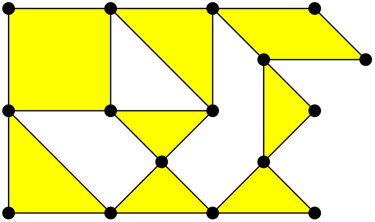

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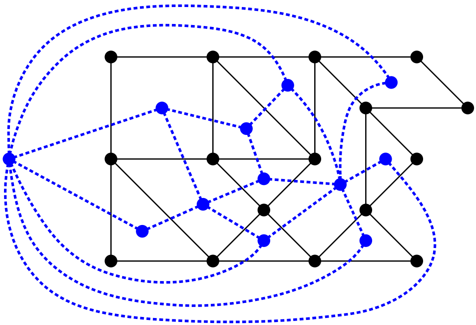

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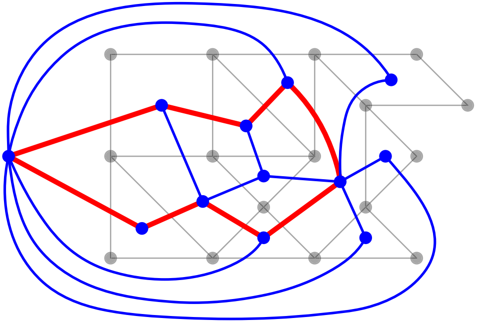

Consider any cycle C in the dual graph.

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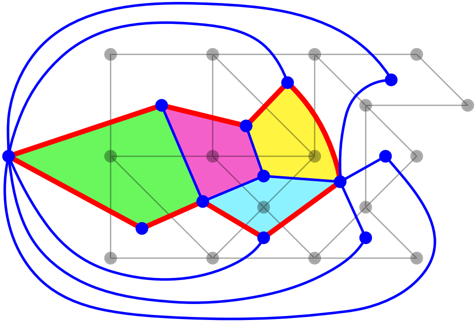

The region bounded by C is the union of faces of the dual graph.

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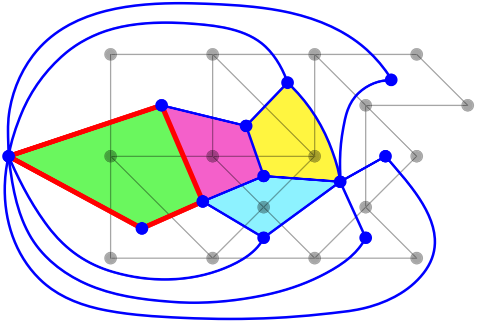

Note that each face has an even number of edges.

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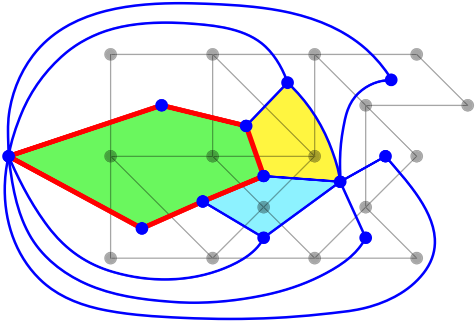

The set of edges of C can be constructed as the *symmetric difference* of the sets of edges of the faces bounded by C.

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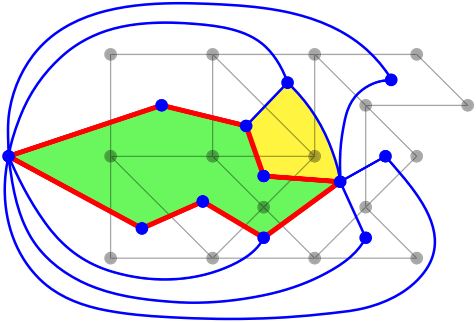

Since the symmetric difference of even-sized sets has even size, we conclude that C has even length. □

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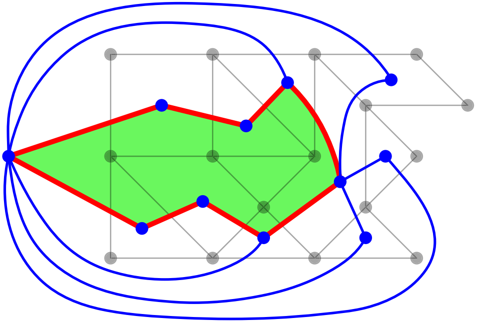

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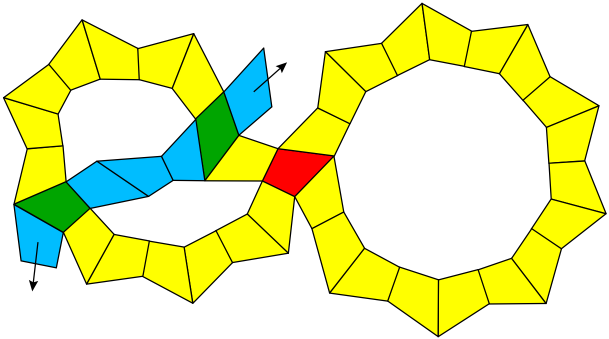
Kite Faces: Zones

Theorem

In a simply connected monohedral polyhedron whose faces are kites, any two zones intersect in an even number of faces.

Proof.



A zone can be turned into a graph whose vertices have degree 2 or 4 (depending on where it self-intersects). Such a graph is Eulerian.

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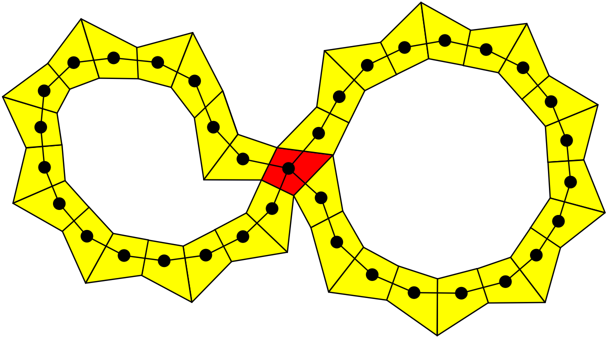

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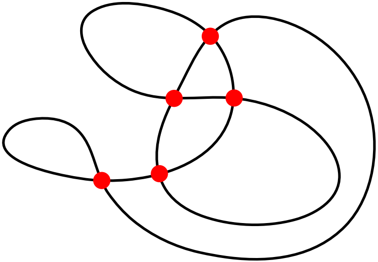

Since this graph is spherical, it is also planar. Thus, by the previous lemma, its dual graph is bipartite.

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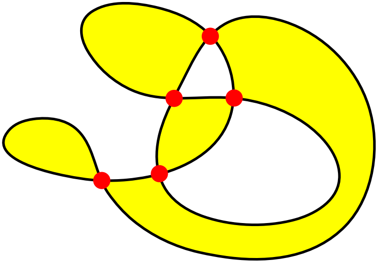

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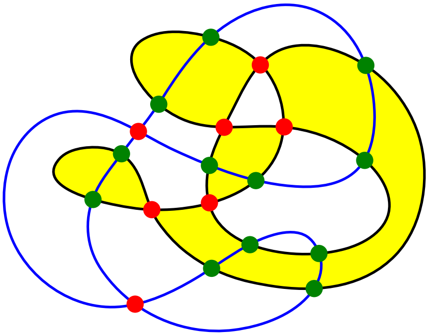


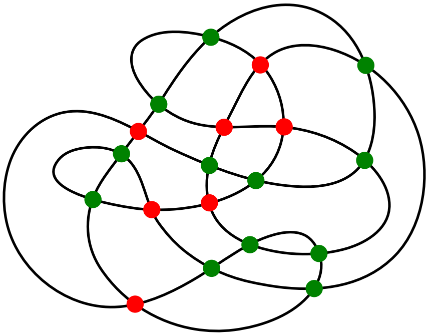
Consider a second zone, which is represented by a closed circuit that may intersect the first one.

Kite Faces: Zones

Theorem

In a simply connected monohedral polyhedron whose faces are kites, any two zones intersect in an even number of faces.

Proof.



Since the two circuits properly intersect each other away from their self-intersections, they intersect an even number of times. □

Conclusions

- If a simply connected monohedral polyhedron has an odd number of faces, then its faces must be **kites**.
- The polyhedron must have at least **two apexes**.
- The polyhedron's zones must self-intersect an **odd** number of times in total.
- Euler's formula and Descartes' theorem are ineffective here, because they are insensitive to the *metric* properties of kites.

Conjecture

In a simply connected monohedral polyhedron whose faces are kites, no zone has self-intersections.

This would imply that all simply connected monohedral polyhedra have an even number of faces.