Minimizing Visible Edges in Polyhedra

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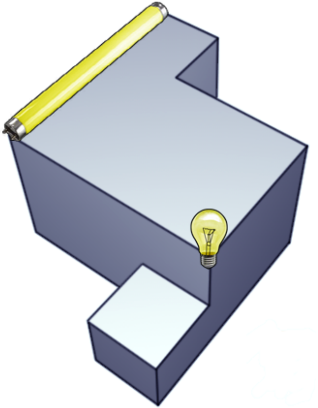
Overview

- 3D Art Gallery Problem
  - Vertex and point guards
  - Edge guards
- Minimizing visible edges in polyhedra
- Spherical Occlusion Diagrams
  - Studying the Swirl Graph
  - Minimizing swirls
  - Minimizing arcs
- Minimizing visible edges for vertex-hidden points

3D Art Gallery Problem

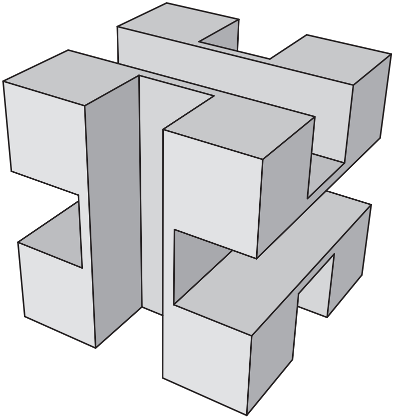
Given a polyhedron in $\mathbb{R}^3$, choose a (preferably small) set of vertices or edges that collectively see its whole interior.



These are called **vertex guards** and **edge guards**.

Vertex-guarding polyhedra

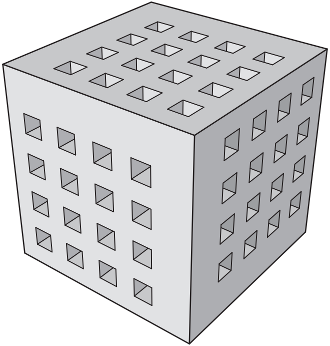
The Art Gallery Problem for *vertex guards* may be <u>unsolvable</u>, even in some orthogonal polyhedra:



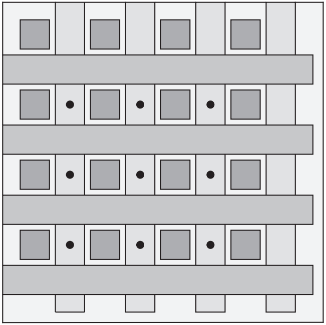
Some points in the central region are invisible to all vertices!

Point-guarding polyhedra

So, we must consider *point guards* that do not lie on vertices.
But there are (orthogonal) polyhedra that require $\Omega(n\sqrt{n})$ guards!



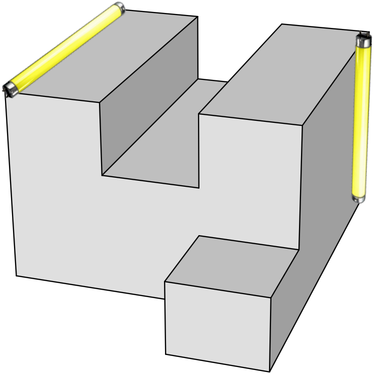
outer view



cross section

Edge-guarding polyhedra

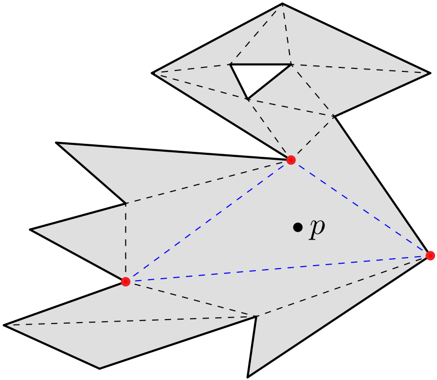
What about edge guards? They are strictly more powerful than point guards: placing a guard on every edge solves the Art Gallery Problem, because each internal point sees at least one edge.



Problem. Does every internal point see at least $c > 1$ edges?

Minimizing visible edges

Consider a cross-section of the polyhedron through any point p, and triangulate it.

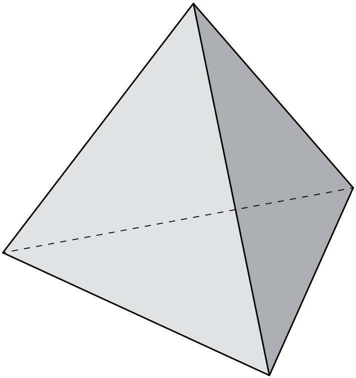


The point p sees at least 3 vertices of the cross section, which correspond to 3 distinct edges of the polyhedron. Hence $c \geq 3$.

Can we do better?

Minimizing visible edges

Note that every point in a tetrahedron sees exactly 6 edges.



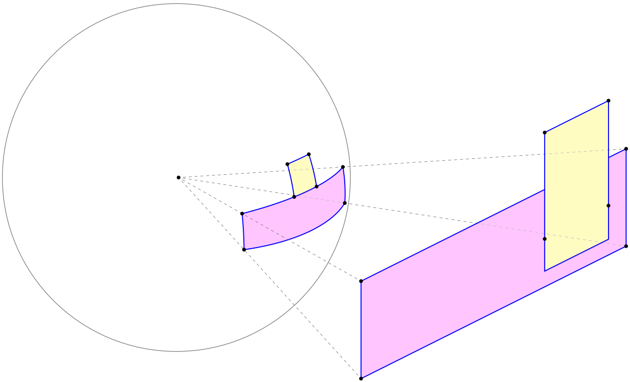
So, c cannot be greater than 6.

Can we prove that $c = 6$?

Minimizing visible edges

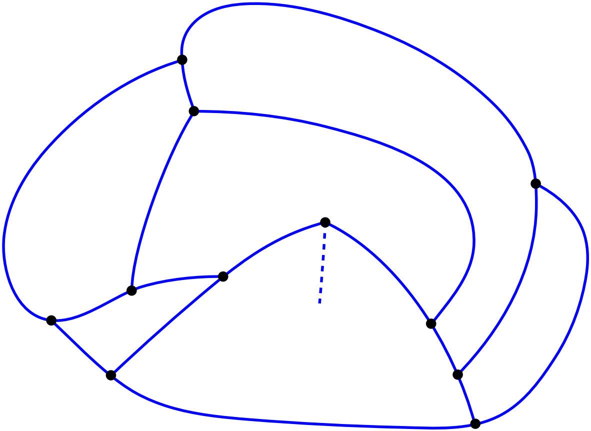
Theorem

In a polyhedron, every point sees at least 6 distinct edges.



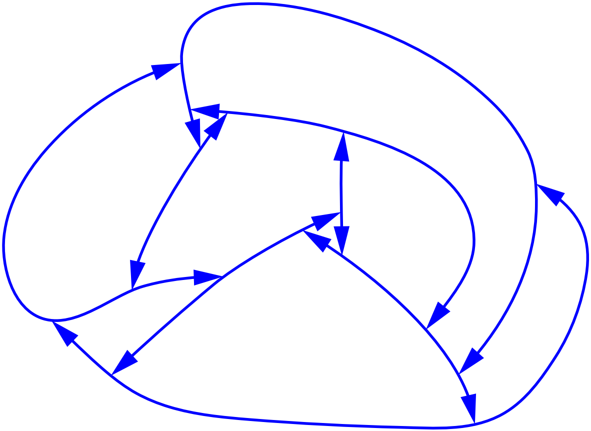
Proof. For any point in a polyhedron, consider the graph obtained by orthographically projecting all visible edge sub-segments onto a small sphere around the point...

Minimizing visible edges



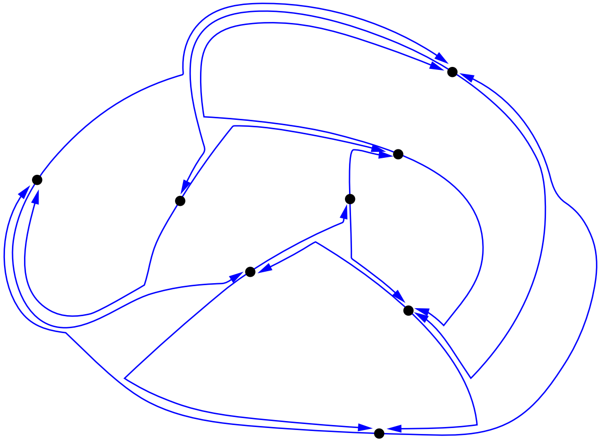
What we obtain is a spherical (hence planar) arrangement of (possibly degenerate) arcs.

Minimizing visible edges



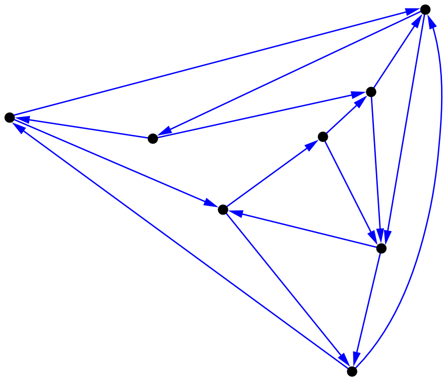
Transform it into a planar arrangement of non-degenerate lines, where each line “feeds into” exactly two other lines.

Minimizing visible edges



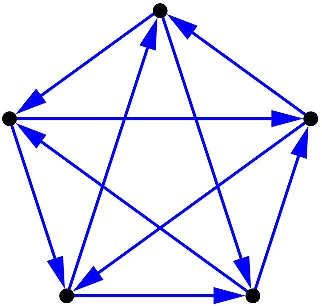
The **contact graph** of this arrangement is a simple planar directed graph where each vertex has outdegree 2.

Minimizing visible edges



If the contact graph has n vertices, it has $2n$ edges. So, $n \geq 5$.
(Indeed, a simple graph on 4 vertices has at most 6 edges, etc.)

Minimizing visible edges

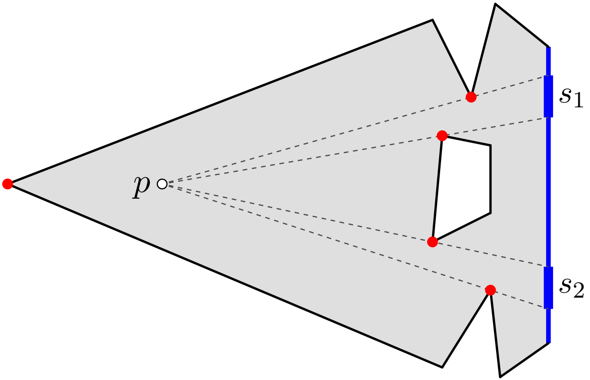


If $n = 5$, the contact graph must be the complete graph K_5, which is not planar (Kuratowski's theorem). Hence $n \geq 6$.

This means that a point in a polyhedron sees at least 6 edge sub-segments.

Minimizing visible edges

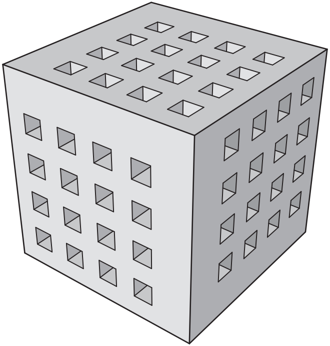
What if a point p sees two sub-segments s_1 and s_2 that are part of the same edge of the polyhedron?



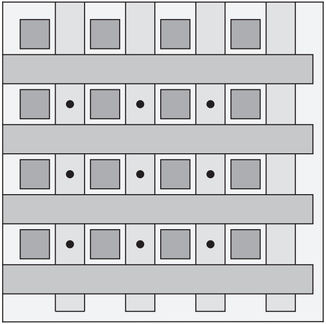
In this case, p sees at least 5 more edges. We conclude that every point in a polyhedron sees at least 6 distinct edges.

Points that see no vertices

What if p does not see any vertex of the polyhedron?



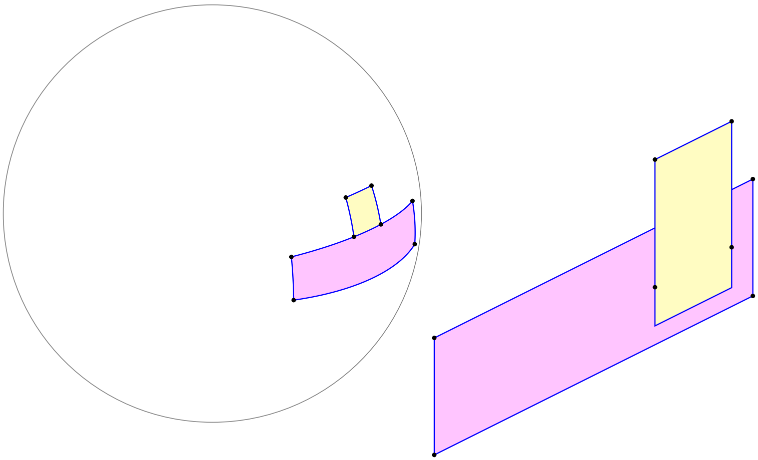
outer view



cross section

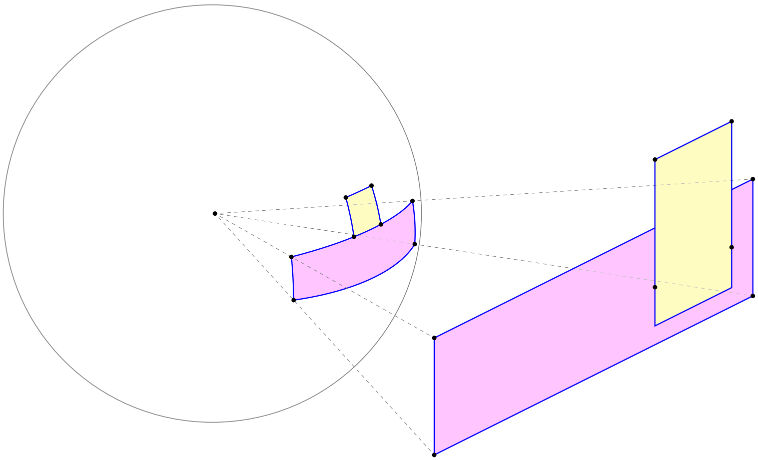
Problem. Can we prove that p necessarily sees more than 6 edges?

Spherical Occlusion Diagrams: introduction



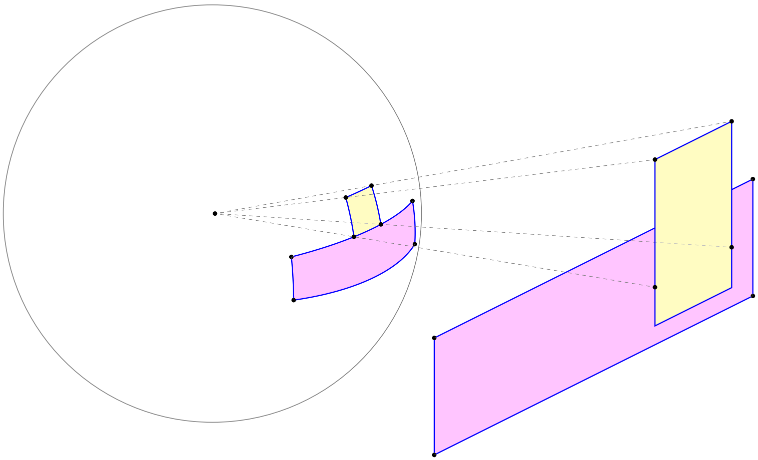
Recall that, when polygons in $\mathbb{R}^3$ are orthographically projected onto a sphere, their edges become arcs of great circle.

Spherical Occlusion Diagrams: introduction



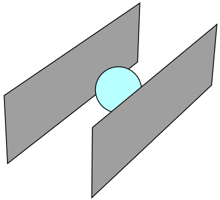
Moreover, when a polygon is partially hidden (i.e., **“occluded”**) by another, in the projection there are arcs feeding into other arcs.

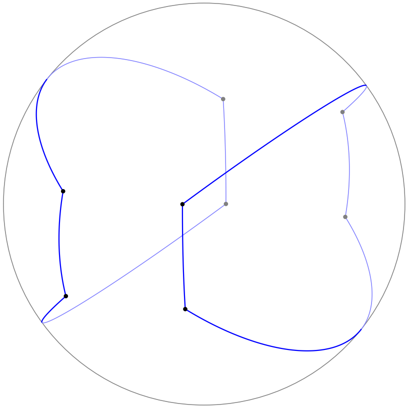
Spherical Occlusion Diagrams: introduction



Moreover, when a polygon is partially hidden (i.e., **"occluded"**) by another, in the projection there are arcs feeding into other arcs.

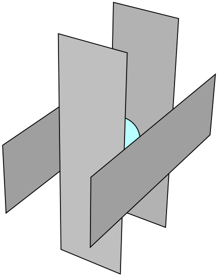
Spherical Occlusion Diagrams: introduction

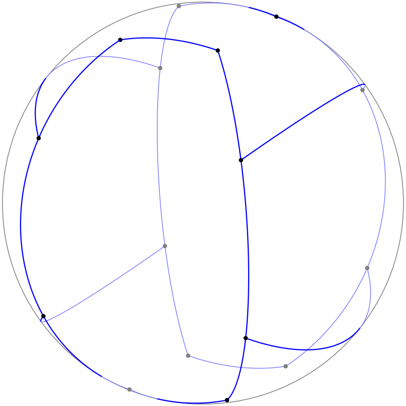




If in an arrangement of polygons all vertices are occluded, then their edges project into a **“Spherical Occlusion Diagram”**.

Spherical Occlusion Diagrams: introduction





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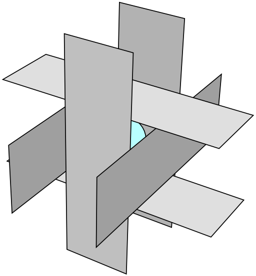
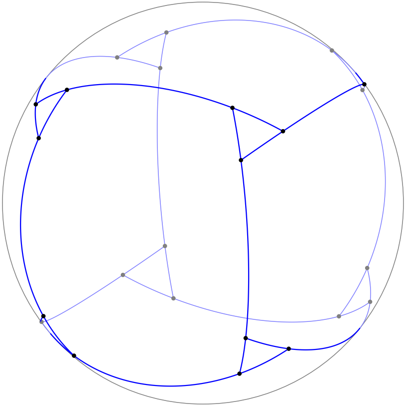
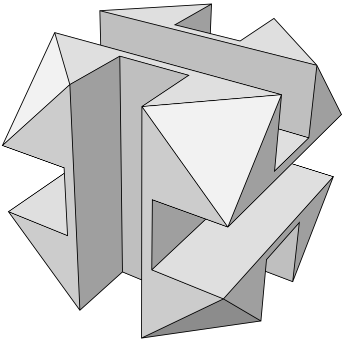
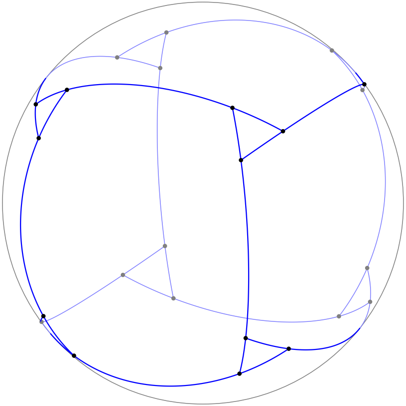
Spherical Occlusion Diagrams: introduction





If in an arrangement of polygons all vertices are occluded, then their edges project into a **"Spherical Occlusion Diagram"**.

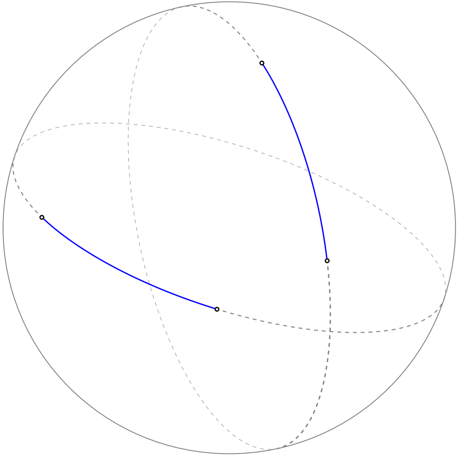
Spherical Occlusion Diagrams: introduction

In particular, this applies to polyhedra: if all vertices are occluded, then the 1-skeleton projects into a Spherical Occlusion Diagram.

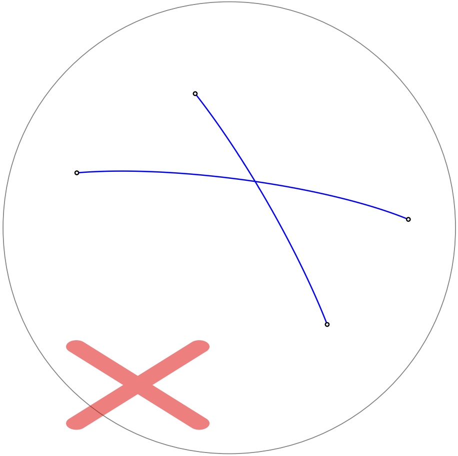
Problem. How many arcs does such a Diagram have at least?

Spherical Occlusion Diagrams: definition



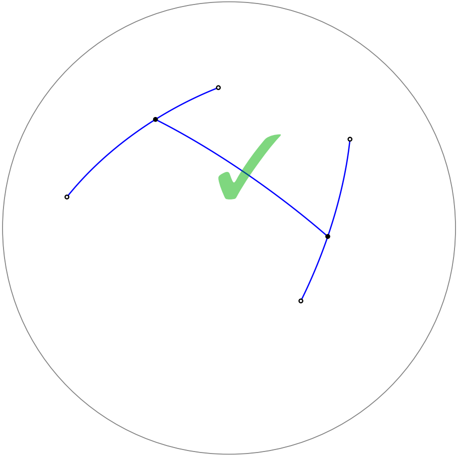
A **Spherical Occlusion Diagram**, or just “Diagram”, is a finite non-empty collection of arcs of great circle on the unit sphere.

Spherical Occlusion Diagrams: definition



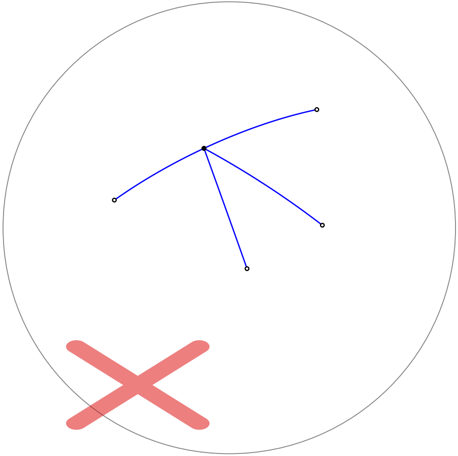
All arcs in a Diagram must be internally disjoint.

Spherical Occlusion Diagrams: definition



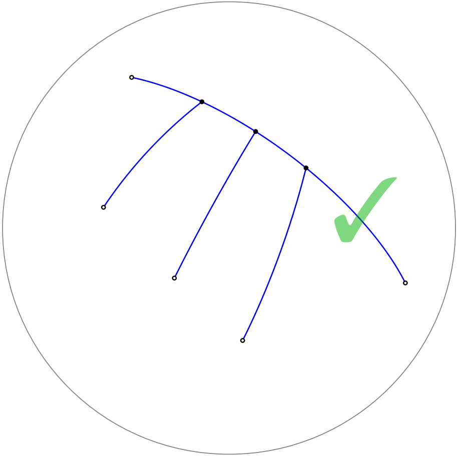
The endpoints of every arc in a Diagram must lie on some other arcs in the Diagram (we say that every arc **“feeds into”** two arcs).

Spherical Occlusion Diagrams: definition



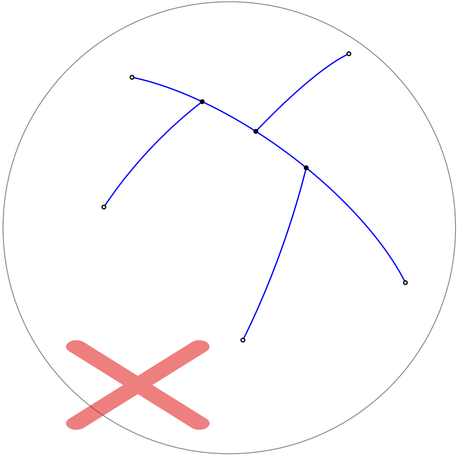
No two arcs in a Diagram can share an endpoint.

Spherical Occlusion Diagrams: definition



All the arcs in a Diagram that feed into the same arc must reach it from the same side.

Spherical Occlusion Diagrams: definition



All the arcs in a Diagram that feed into the same arc must reach it from the same side.

Spherical Occlusion Diagrams: axioms

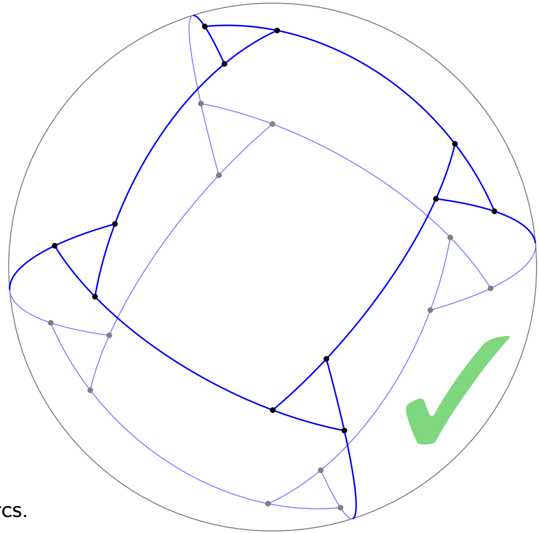


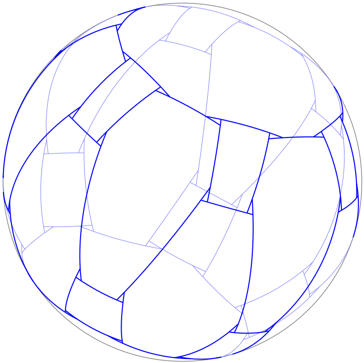
Diagram axioms:

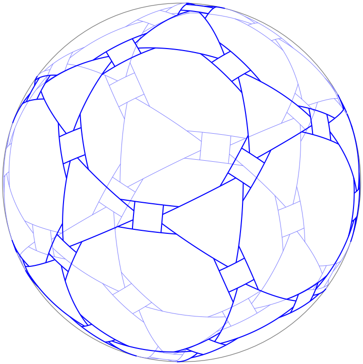
1. If two arcs intersect,
   one feeds into the other.
2. Each arc feeds into two arcs.
3. All arcs that feed into the same arc
   reach it from the same side.

Spherical Occlusion Diagrams: basic properties

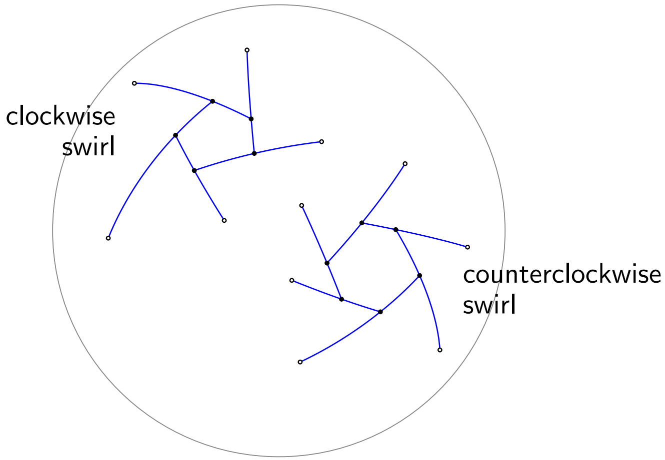
Theorem

- *No two arcs in a Diagram feed into each other.*
- *Each arc in a Diagram is shorter than a great semicircle.*
- *Every Diagram is connected.*
- *A Diagram with n arcs partitions the sphere into $n + 2$ spherically convex regions (called* **"tiles"** *).*

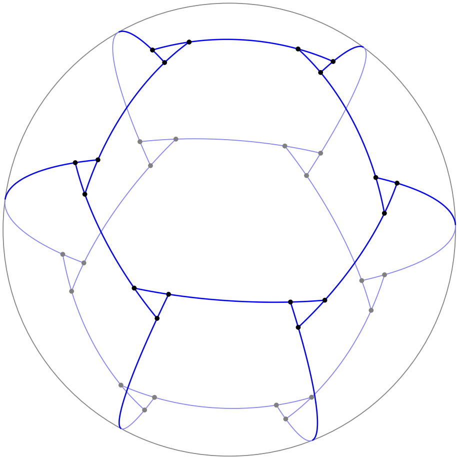




Spherical Occlusion Diagrams: swirls

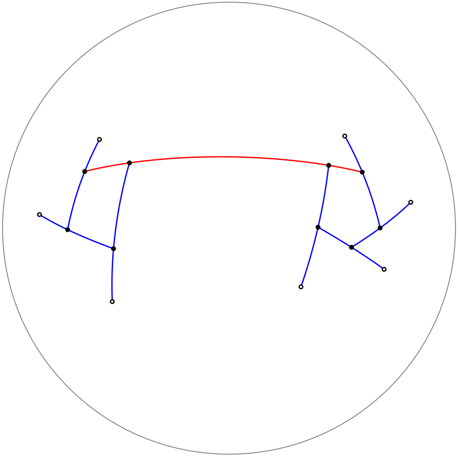


A **swirl** in a Diagram is a cycle of arcs such that each arc feeds into the next going clockwise or counterclockwise.



A Diagram is **swirling** if every arc is part of two swirls.

Swirl Graph

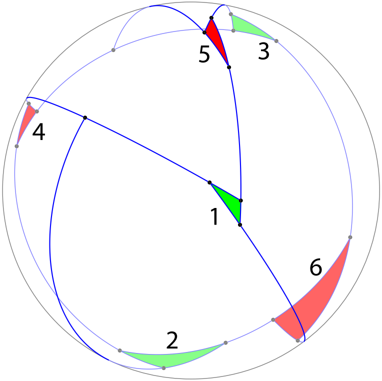


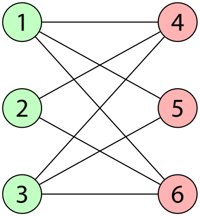
The **Swirl Graph** of a Diagram is an undirected multigraph on the set of swirls. For each arc shared by two swirls, there is an edge in the Swirl Graph.

Swirl Graph

Theorem

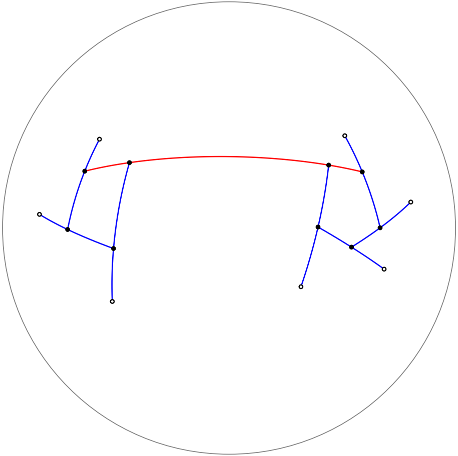
The Swirl Graph of a Diagram is a simple planar bipartite graph with non-empty partite sets.





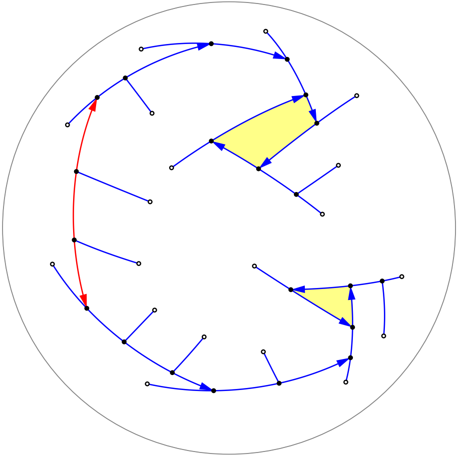
Proof. Obviously the Swirl Graph is spherical, hence planar. The bipartition is given by the clockwise and counterclockwise swirls...

Swirl Graph



Each edge in the Swirl Graph must connect a clockwise and a counterclockwise swirl. So the Swirl Graph is bipartite.

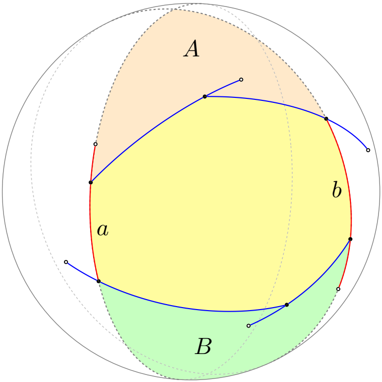
Swirl Graph



To find a (counter)clockwise swirl, start anywhere and follow the Diagram (counter)clockwise. Hence the partite sets are not empty.

Swirl Graph

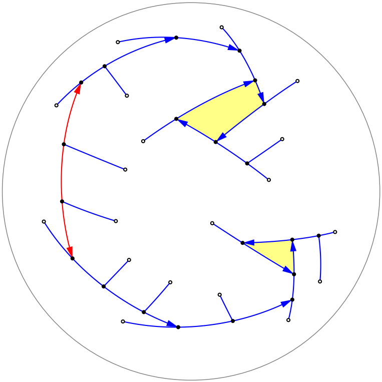
Assume that the yellow swirl shares arcs a and b with another swirl. The second swirl must be located in the highlighted spherical lune.



Since a goes upward, the second swirl must be in A. But b goes downward, so the second swirl must be in B: contradiction. Hence, the Swirl Graph is simple.

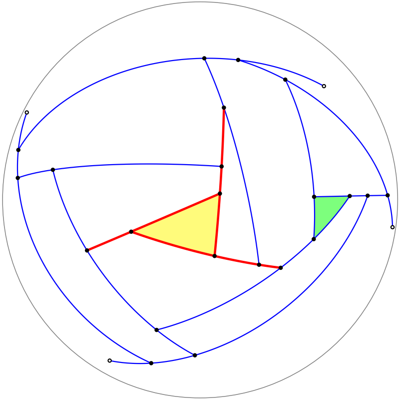
More swirls?

We want to prove that a Diagram has more than two swirls.
Can we expand on our previous idea?



Maybe following arcs in other directions leads to different swirls?
Unfortunately, most reasonable ideas are bound to fail...

Counterexample

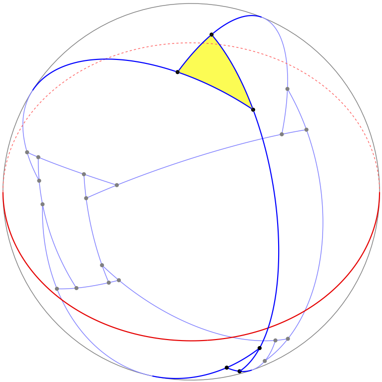


- Starting from any of the red arcs and going (counter)clockwise in the four possible ways ends up in the same two swirls.
- Taking any edge of the counterclockwise swirl and following the Diagram counterclockwise away from the swirl ends up back in the counterclockwise swirl.

Counterexample

Another reasonable conjecture is that every hemisphere intersects both a clockwise and a counterclockwise swirl.

Unfortunately, there is a counterexample:

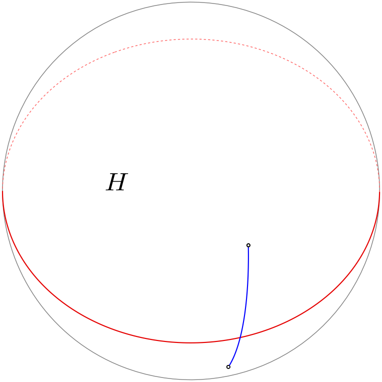


But we can prove that *every hemisphere contains at least one swirl!*

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

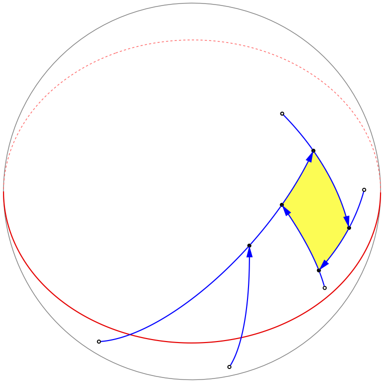


Proof. Take any hemisphere H. Since the Diagram is connected and tiles are convex, there is an arc crossing the boundary of H.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

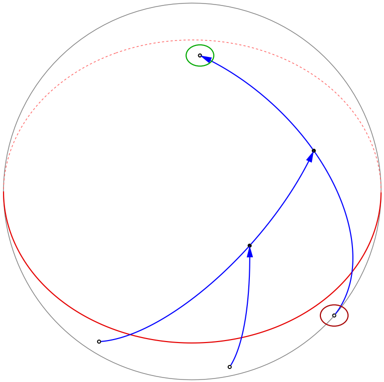


Follow the Diagram clockwise starting from this arc. If we remain in H, we eventually find a clockwise swirl fully contained in H.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

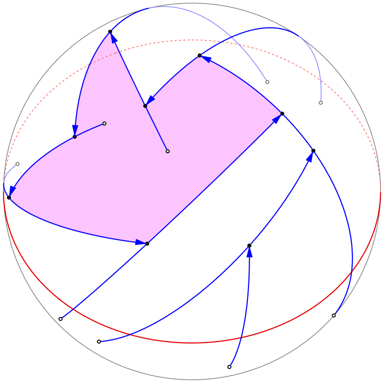


Otherwise, we find one arc whose clockwise endpoint is outside H.
But then, the other endpoint is in H. Reach that endpoint.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

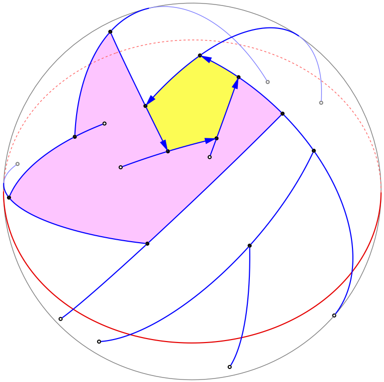


Continuing in this fashion, we either find a clockwise swirl in H, or eventually we enclose a region within H by going counterclockwise.

Hemisphere lemma

Lemma

In a Diagram, every hemisphere contains at least one full swirl.

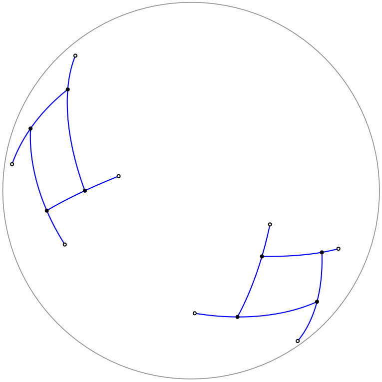


In this case, starting from the boundary of the enclosed region and following the Diagram counterclockwise, we eventually find a swirl.

More swirls

Corollary

Every Diagram has at least 4 swirls.



Proof. We already know that a Diagram has 2 swirls.

More swirls

Corollary

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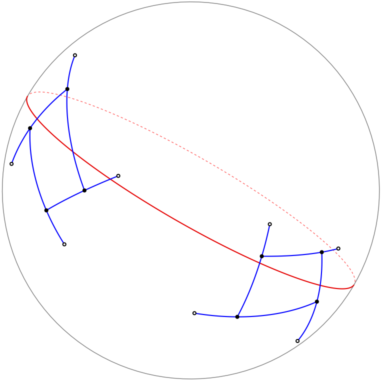
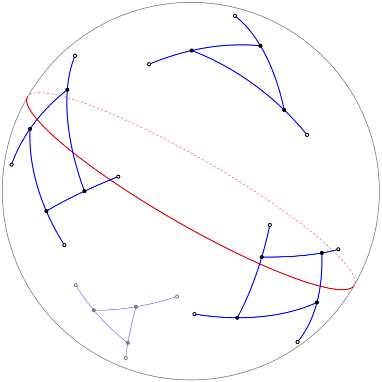


Take a great circle that properly intersects both swirls.

More swirls

Corollary

Every Diagram has at least 4 swirls.

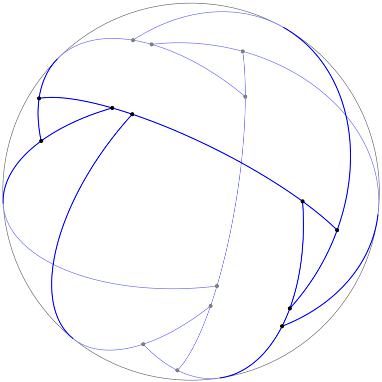


By the previous lemma, each hemisphere contains one new swirl.

Minimizing arcs (and swirls)

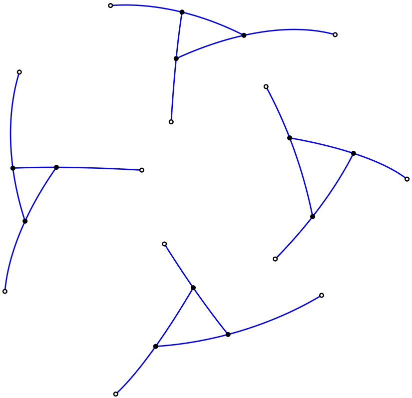
Theorem

Every Diagram has at least 8 arcs, and there exist Diagrams with exactly 8 arcs (and exactly 4 swirls).



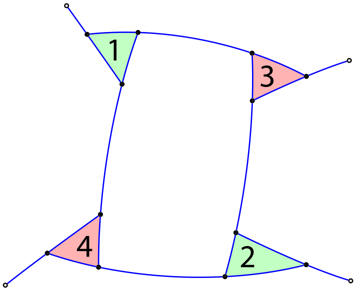
Proof. This is an example of a Diagram with 8 arcs and 4 swirls...

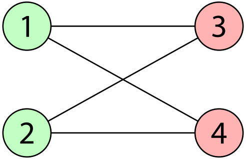
Minimizing arcs (and swirls)



We know that a Diagram has at least 4 swirls. Obviously, if they do not share any arcs, then the Diagram has at least 12 arcs.

Minimizing arcs (and swirls)





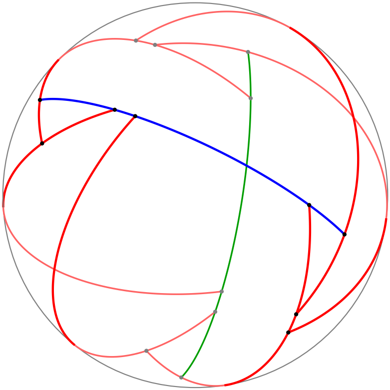
Since the Swirl Graph must be simple and bipartite, these 4 swirls can share at most 4 arcs. Thus the Diagram has at least 8 arcs.

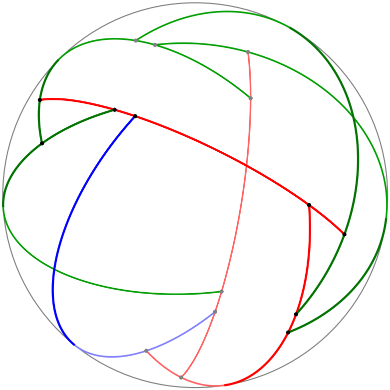
Edges are distinct

We can prove that the 8-arc configuration is unique.

Also, note that the contact graph has diameter 2.

(In other words, given any arc, each other arc is either a neighbor or a neighbor of a neighbor.)





It follows that no two arcs are aligned, and thus they correspond to distinct visible edges of a polyhedron.

Disproving old conjectures

Conjecture

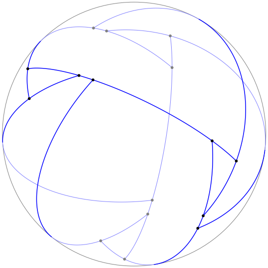
There are no Diagrams with fewer than 12 arcs.

False! There are Diagrams with 8, 9, 10, 11 arcs.

Conjecture

Any Diagram can be constructed by a sequence of "elementary operations" starting from a swirling Diagram (e.g., continuously shifting arcs' endpoints or adding arcs).

False! An 8-arc Diagram cannot be obtained from a swirling one.



Conclusion and future work

- We proved that a point in a polyhedron sees at least 6 edges.
- If the point does not see any vertices, it sees at least 8 edges.

New problem. How many *faces* does a point see at least?

The unrestricted case is easy:

Theorem (Lusternik-Schnirelmann, 1930)

If the n-dimensional sphere is covered by $n+1$ closed sets, one of them contains a pair of antipodal points.

Corollary

In a polyhedron, every point sees at least 4 faces.
(The tetrahedron gives a matching lower bound.)

However, for points that see no vertices, we still have no answer.

Conjecture

In a polyhedron, if a point sees no vertices, it sees at least 8 faces.