Chasing Puppies

Mobile Beacon Routing on Closed Curves

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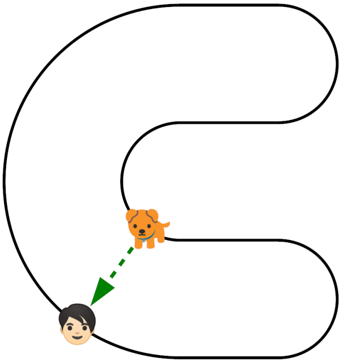
Chasing puppies on simple closed tracks

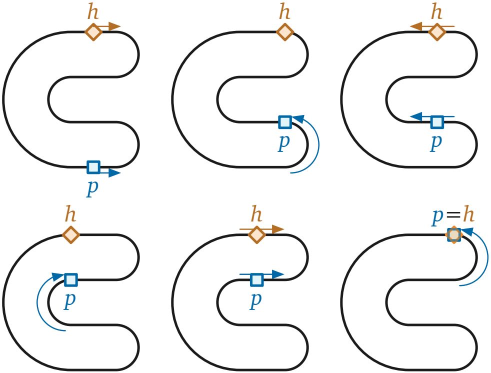
A **human** and a **puppy** are constrained to move on a simple closed curve. The puppy is pulled by a leash, and always tries to locally minimize its distance to the human. The puppy's speed is infinite.



Problem. Can the human always catch the puppy?
(Posed by Michael Biro at the open-problem session of CCCG'13)

Chasing puppies: Example

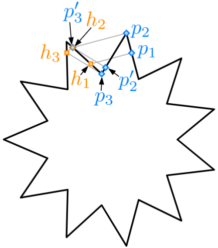
A possible strategy for catching the puppy on this track:

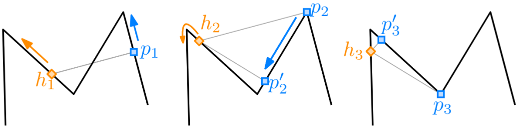


It seems that moving steadily in any direction (say, clockwise) is always going to be successful. But there are counterexamples...

Chasing puppies: Counterexamples

There are tracks where the human fails to catch the puppy if he goes in the wrong direction:

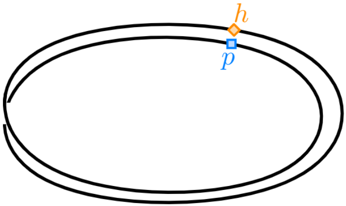




However, in this example going the other way obviously succeeds.

Chasing puppies: Counterexamples

There are also (self-intersecting) tracks where the human and the puppy may never meet:



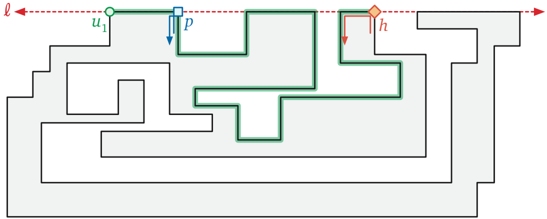
(With self-intersecting tracks, the two strands at an intersection point are considered to be distinct points of the curve.)

So the assumption that the track be simple is essential.

Catching puppies on orthogonal polygons

Theorem

If the track is an **orthogonal polygon**, *the human will catch the puppy by going in any direction around the polygon at most twice.*

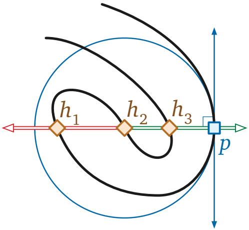


Proof. Say the human is walking counterclockwise. Consider the top-left vertex u_1. It takes at most one loop for the human to reach u_1 for the first time. After that, the puppy will always be found between the human and u_1 (because it cannot escape the top "pocket"), and so it will be caught within another loop. □

Smooth tracks: Critical configurations

Suppose now the track is a *smooth* simple closed curve.[1]

A configuration is **critical** if the line normal to the curve at the puppy's location contains the human.



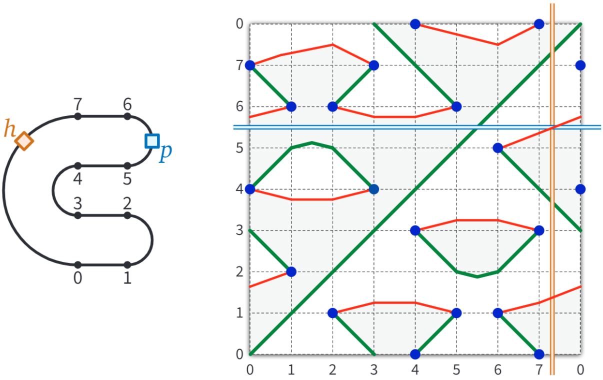
We distinguish three types of critical configurations based on the center of curvature c of the curve at the puppy's location:

- **Stable:** if the human is between c and the puppy (e.g., h_3),
- **Unstable:** if c is between the human and the puppy (e.g., h_1),
- **Pivot:** if the human is exactly on c (e.g., h_2),

[1] Technically, we need the curve to have continuous third derivative.

Attraction diagram (aka "puppy diagram")

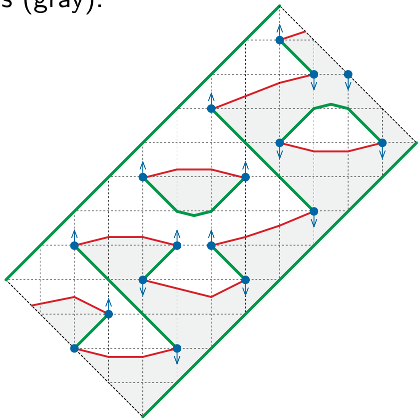
We can view the configuration space as a *torus*, where we highlight stable, unstable, and pivot configurations:



This is called the **Attraction diagram** or *puppy diagram*.

Puppy diagram: Critical curves

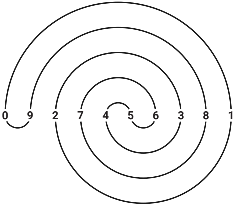
The critical configurations form disjoint closed curves in the configuration space, separating the regions where the puppy moves forward along the curve (white) from the regions where the puppy moves backwards (gray).

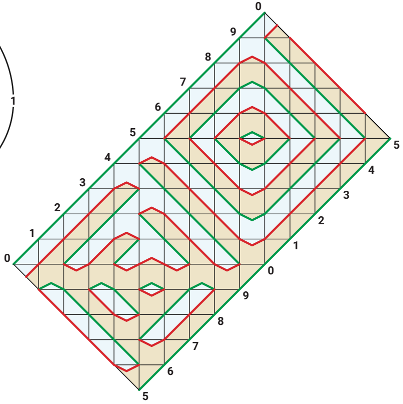


A critical curve is **essential** if it cannot be continuously contracted to a point. The main diagonal is always an essential critical curve.

Puppy diagram: Critical curves

Another example of a track and its puppy diagram:





We will prove that the puppy diagram of a simple and smooth track always has exactly two essential critical curves.

Puppy diagram: Critical curves

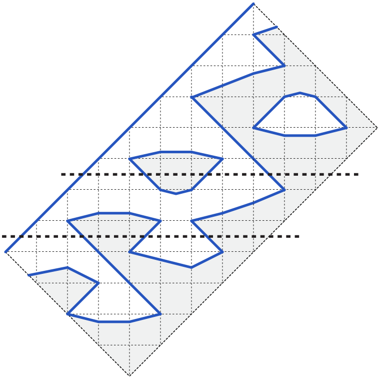
Lemma

The puppy diagram of any closed curve (simple or not) has an <u>even number</u> of essential critical curves.

Proof. Any horizontal line intersects each non-essential critical curve an <u>even</u> number of times.

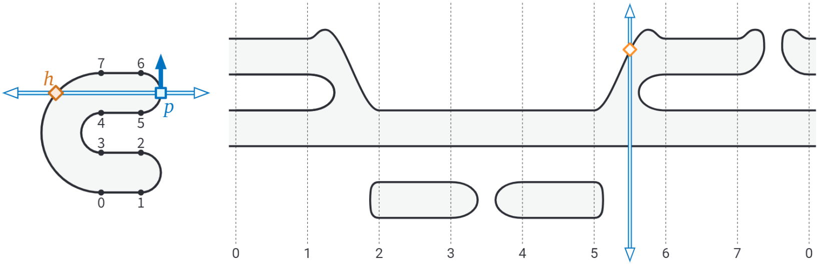
Since a horizontal line intersects the main diagonal exactly once, it must intersect each essential critical curve an <u>odd</u> number of times.

Also, since critical curves separate white and gray regions, the total number of their intersections with any horizontal line is <u>even</u>. □



Dual attraction diagram (aka "human diagram")

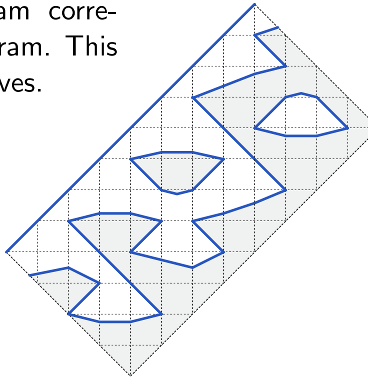
We can also view the configuration space as an *infinite cylinder*, where each vertical line represents a normal to the curve at a given point p. The intersections between the normal and the curve are the possible positions of the human when the puppy is at p.



When we trace out the curve along every normal, we obtain the **Dual attraction diagram** or *human diagram*.

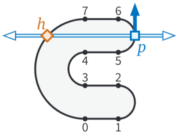
Human and puppy diagram: Correspondence

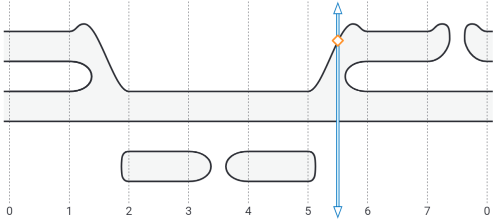
Each horizontal line in the puppy diagram corresponds to a vertical line in the human diagram. This induces a continuous mapping between curves.



Observation

The puppy diagram and the human diagram have the same number of (non-)essential critical curves.



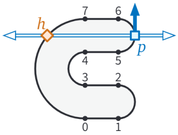


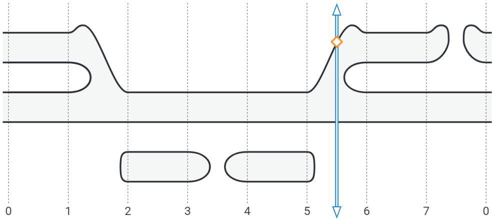
Human diagram: Critical curves

Lemma

The human diagram of a simple closed curve has exactly two essential critical curves.

Proof. Since the curve is simple, no two critical curves intersect each other. So, the *essential* ones have a well-defined total vertical order. Also, each essential critical curve in the puppy diagram goes through every possible human position on the curve. In particular, it goes through a point on the convex hull. Thus, the corresponding curve in the human diagram is an *extremal* one: either the topmost or the bottom-most. Hence, there can be at most two essential critical curves. □

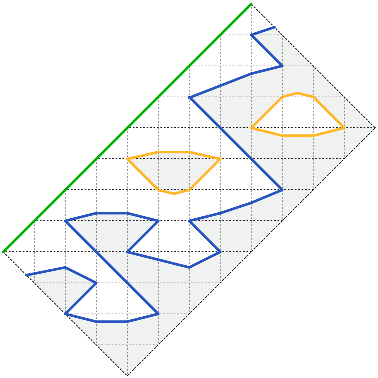




Puppy diagram: River

Corollary

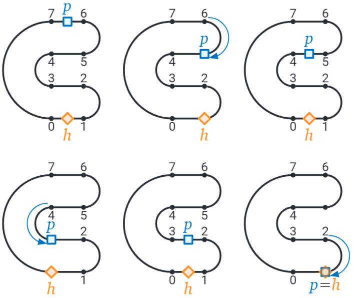
The puppy diagram of a simple closed curve has exactly two essential critical curves, as well.

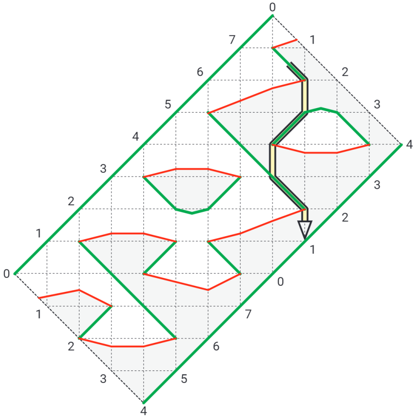


As we know, one essential critical curve is the main diagonal.
The other essential critical curve is called the river.

Dexter and sinister strategies

Any *strategy* to catch the puppy corresponds to a *path* in the puppy diagram consisting of stable critical paths and vertical (upward or downward) segments terminating on the main diagonal.

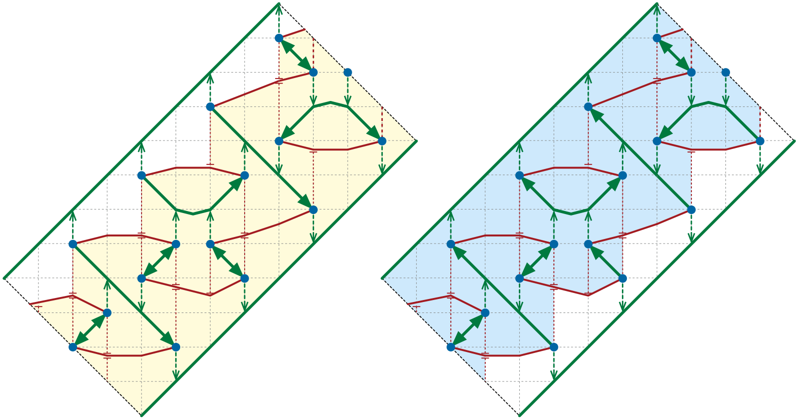




If the main diagonal is reached from above, the strategy is called **dexter**. Otherwise, it is called **sinister**.

Dexter and sinister configurations

Also, a *configuration* is called **dexter** (resp. **sinister**) if there is a dexter (resp. sinister) strategy starting in that configuration.



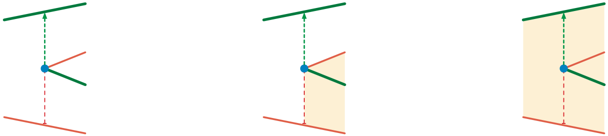
We will prove that dexter and sinister configurations collectively cover the whole configuration space.

Dexter configurations

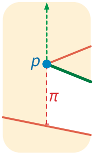
Lemma

The dexter configurations form a simply connected set containing (an upper neighborhood of) the main diagonal and the river.

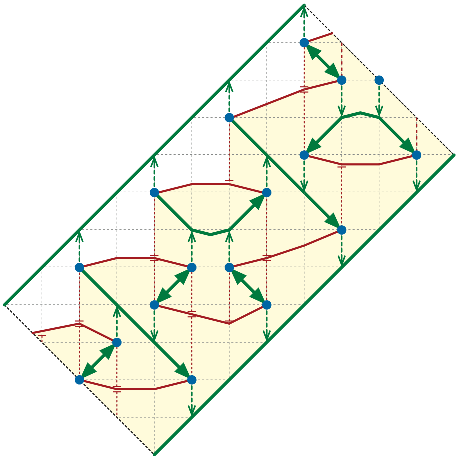
Proof. Partition the configuration space into "trapezoids" by drawing vertical segments at the pivots. The trapezoids incident to the same stable path are either all dexter or all non-dexter. So, there are only three possible configurations for the trapezoids around a pivot:



Also, the dexter set cannot have holes, or it would form a "bracket":



Hence, the dexter set is simply connected, and is bounded below by the main diagonal and above by unstable critical paths and upward pivots. It is easy to see that such a set must contain the river. □

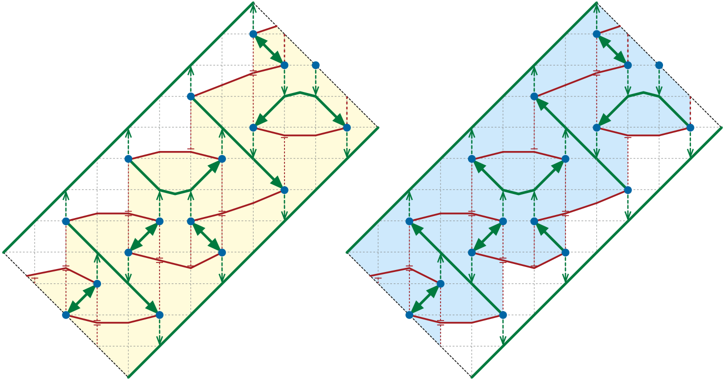


Catching puppies on simple smooth tracks

Theorem

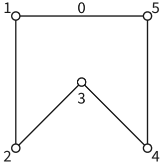
If the track is a **simple smooth curve***, the human has a strategy to catch the puppy from every configuration.*

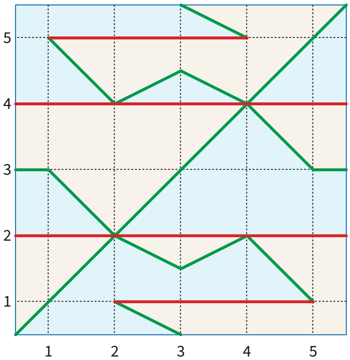
Proof. The dexter (resp. sinister) set includes all configurations below (resp. above) the river. So, for each configuration there is either a dexter or a sinister strategy (possibly both). □



Chasing puppies on simple polygons

If the track is a *polygon*, it does not have well-defined tangents at its vertices. As it turns out, the critical curves in its puppy diagram are not necessarily disjoint, and may be *degenerate*:

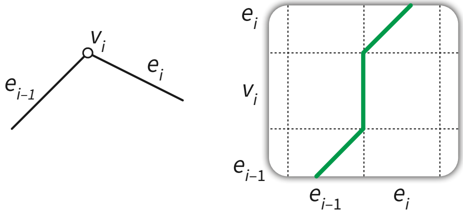




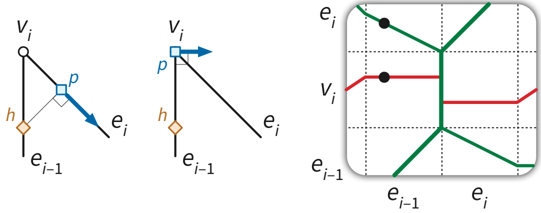
We will sketch a proof that the previous theorem can be extended to simple polygonal tracks.

Extended puppy diagram

To properly describe the puppy's behavior, we must account for its *angular motion* at the vertices. So we extend the configuration space at the vertices by including the *direction* the puppy is facing:

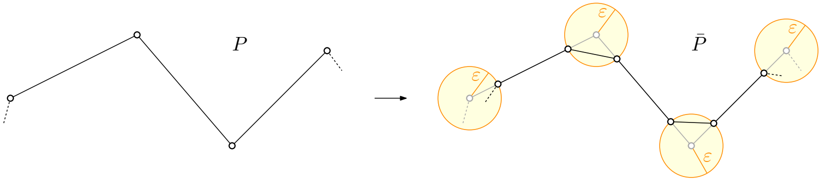


Acute angles are responsible for *degeneracies* in the puppy diagram:

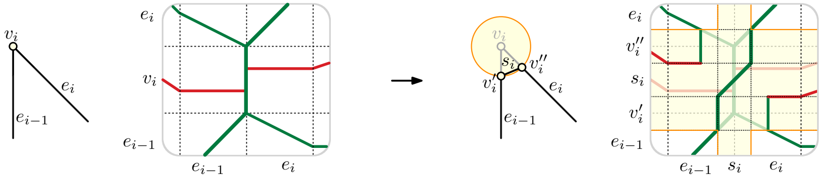


Smoothing out corners

In order to eliminate these degeneracies, we smooth out the polygon by "cutting away" its corners:

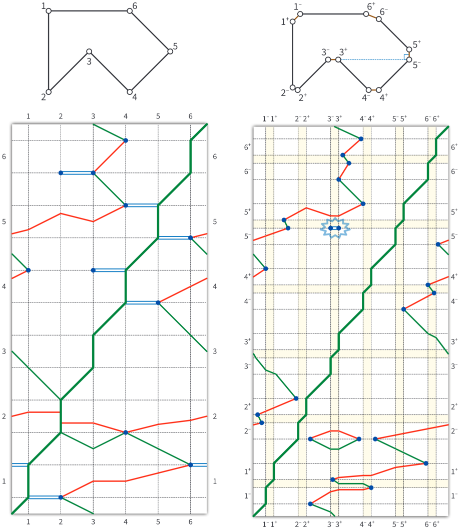


This way, each *acute* angle becomes a pair of *obtuse* angles, and the degeneracies disappear:



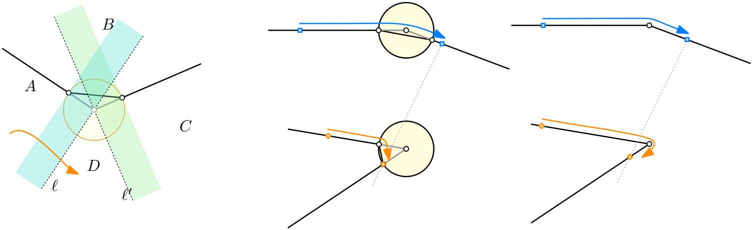
Catching puppies on smooth polygons

The resulting polygon has all the relevant properties of smooth curves. In particular, its puppy diagram has exactly <u>two essential critical curves</u>.



Catching puppies on simple polygons

So, if we "smooth out" the polygon, the human has a strategy to catch the puppy. This strategy can always be converted back into a strategy to catch the puppy in the original "unsmoothed" polygon.



Theorem

If the track is a **simple polygon**, *the human has a strategy to catch the puppy from every configuration.*

Conclusions and open problems

We have shown that the human can always catch the puppy if the track is a simple smooth curve or a simple polygon.

Here are some directions for future research:

- Is it always possible to catch the puppy by walking only clockwise or only counterclockwise?
- What if the track is self-intersecting? By "doubling up" a closed curve multiple times, we can construct counterexamples of any turning number other than ± 1. We conjecture that all self-intersecting tracks with turning number ± 1 have exactly two essential critical curves, and hence allow the human to catch the puppy.
- What if puppy and human can move *inside* the polygon? If the polygon has no holes, the human has an easy strategy to catch the puppy. If there are holes, the problem is at least as hard as the one for simple polygons (consider a thin polygon with one large hole).
- What if the human wants to *avoid* a rabid puppy? Under what conditions can he reach all points on the track without ever meeting the puppy?