A Theory of Spherical Diagrams

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(work in progress...)

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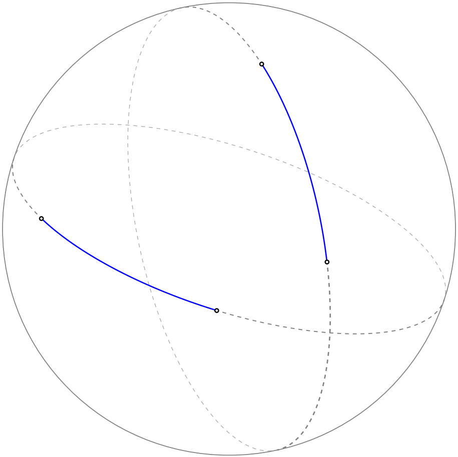
Overview

Spherical Occlusion Diagrams

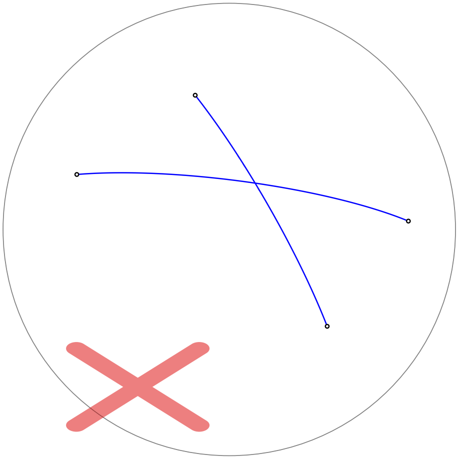
- Definition
- Examples
- Basic properties
- Swirls and uniformity
- Re-interpretation

Spherical Occlusion Diagrams: definition



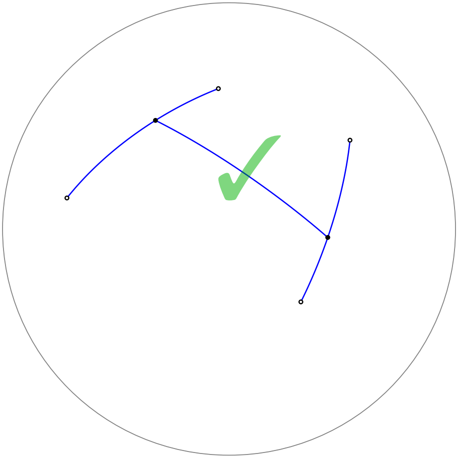
A **Spherical Occlusion Diagram**, or just "Diagram", is a finite non-empty collection of arcs of great circle on the unit sphere.

Spherical Occlusion Diagrams: definition



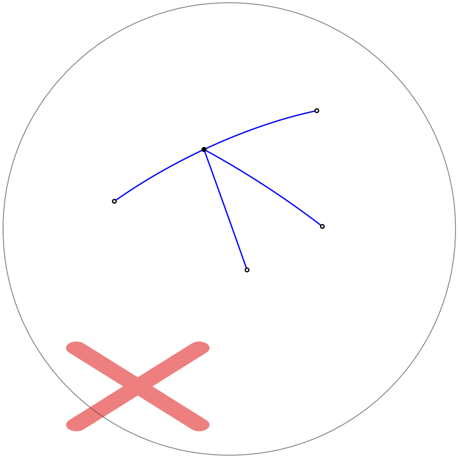
All arcs in a Diagram must be internally disjoint.

Spherical Occlusion Diagrams: definition



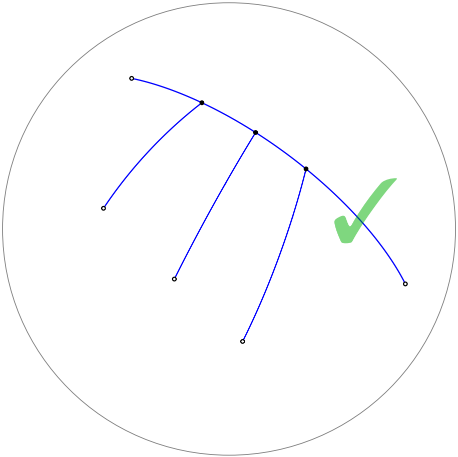
The endpoints of every arc in a Diagram must lie on some other arcs in the Diagram (we say that every arc **"feeds into"** two arcs).

Spherical Occlusion Diagrams: definition



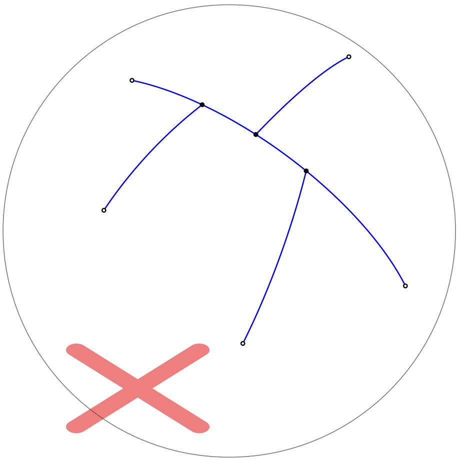
No two arcs in a Diagram can share an endpoint.

Spherical Occlusion Diagrams: definition



All the arcs in a Diagram that feed into the same arc must reach it from the same side.

Spherical Occlusion Diagrams: definition



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Spherical Occlusion Diagrams: examples

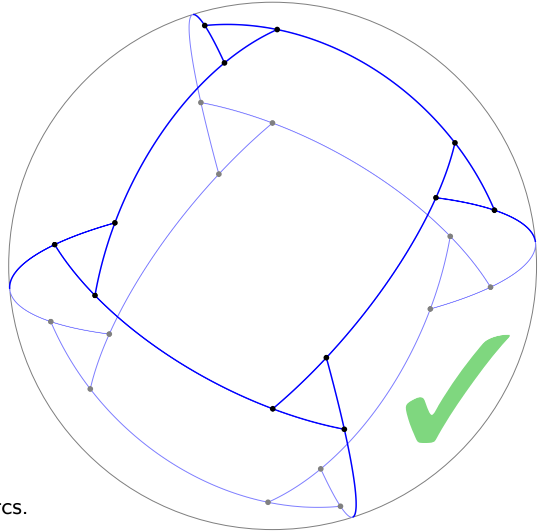
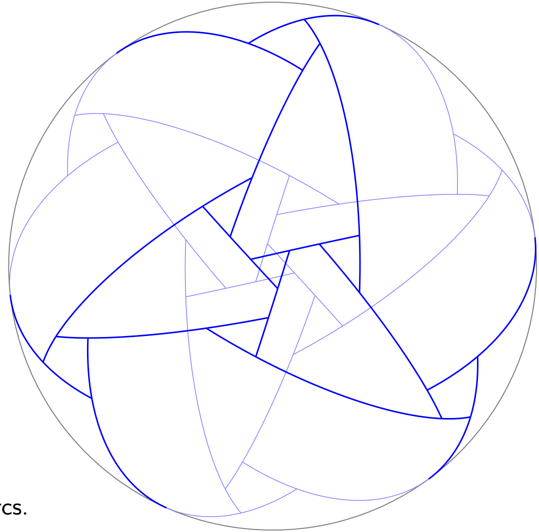


Diagram axioms:

1. If two arcs intersect, one feeds into the other.
2. Each arc feeds into two arcs.
3. All arcs that feed into the same arc reach it from the same side.

Spherical Occlusion Diagrams: examples



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Spherical Occlusion Diagrams: examples

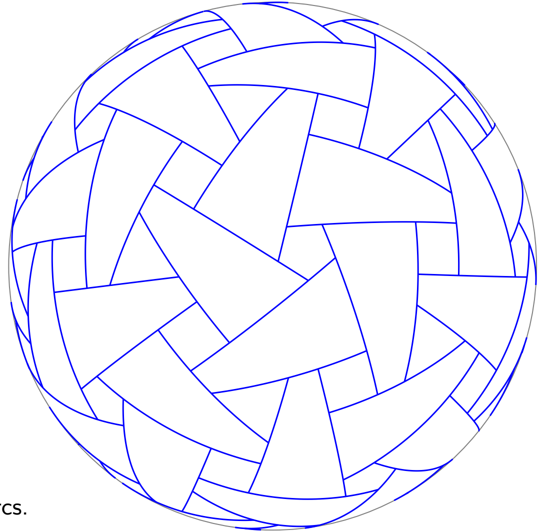

Diagram axioms:

1. If two arcs intersect,
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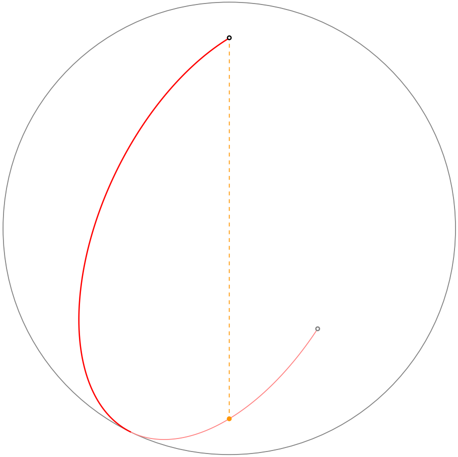
2. Each arc feeds into two arcs.
3. All arcs that feed into the same arc
   reach it from the same side.

Spherical Occlusion Diagrams: basic properties

Proposition

Every arc in a Diagram is strictly shorter than a great semicircle.



Proof. Otherwise it would have arcs feeding into it from both sides.

Spherical Occlusion Diagrams: basic properties

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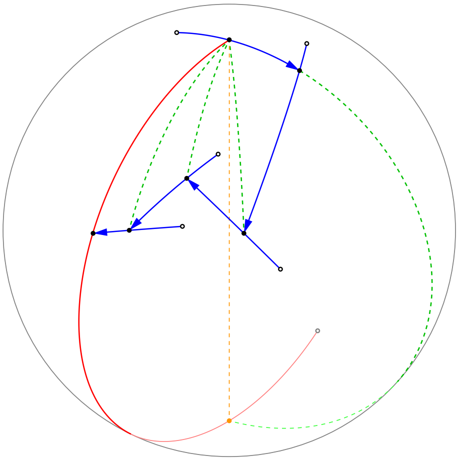

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Spherical Occlusion Diagrams: basic properties

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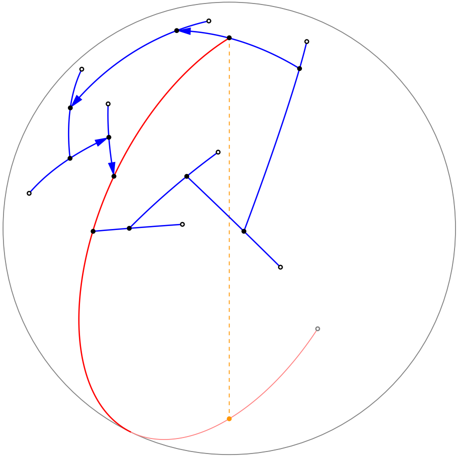
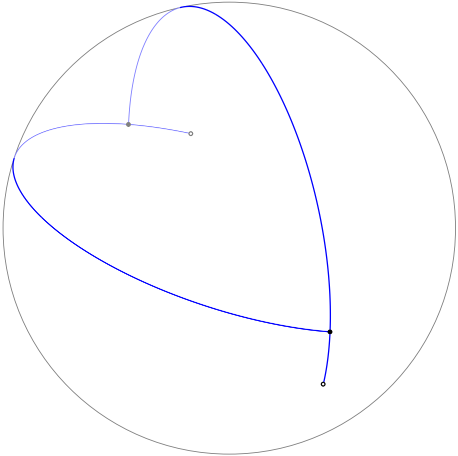


Proof. Otherwise it would have arcs feeding into it from both sides.

Spherical Occlusion Diagrams: basic properties

Corollary

No two arcs in a Diagram feed into each other.

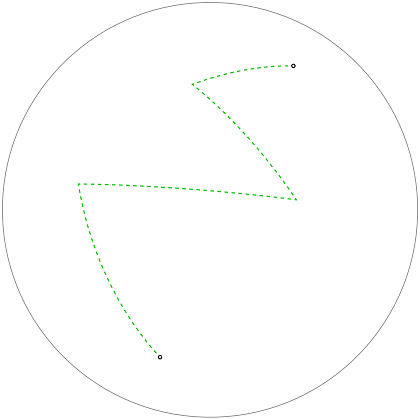


Proof. Otherwise they would be longer than a great semicircle.

Spherical Occlusion Diagrams: basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

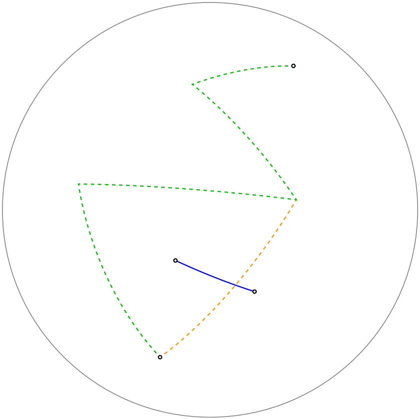


Proof. Two points in the same region can be connected by a chain of arcs of great circle that does not intersect the Diagram.

Spherical Occlusion Diagrams: basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **“tiles”** *).*

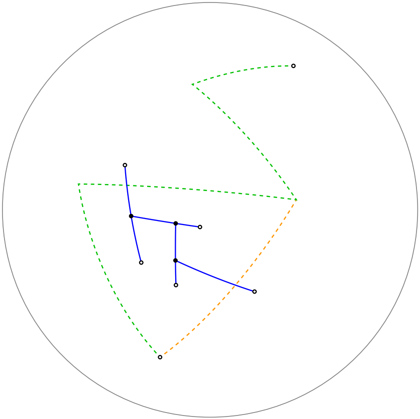


The arc joining the first and the third vertex of the chain does not intersect the Diagram, either...

Spherical Occlusion Diagrams: basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

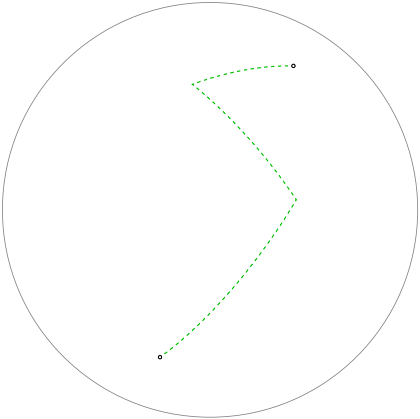


...Otherwise, following the Diagram we would intersect the first two arcs in the chain, which is impossible by assumption.

Spherical Occlusion Diagrams: basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

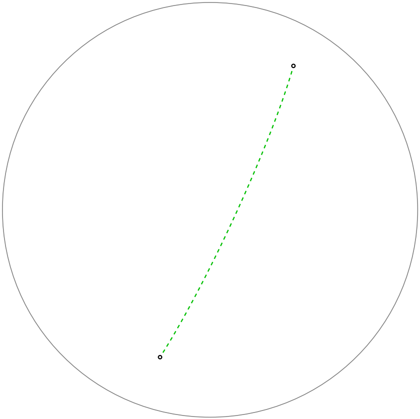


So we can simplify the chain, reducing it by one arc. Inductively repeating this reasoning, we can reduce the chain to a single arc.

Spherical Occlusion Diagrams: basic properties

Proposition

A Diagram partitions the sphere into convex regions (or **"tiles"** *).*

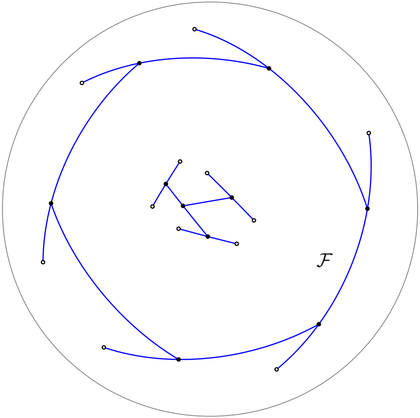


Since any two points in the region are connected by an arc of great circle that does not intersect the Diagram, the region is convex.

Spherical Occlusion Diagrams: basic properties

Corollary

Every Diagram is connected.

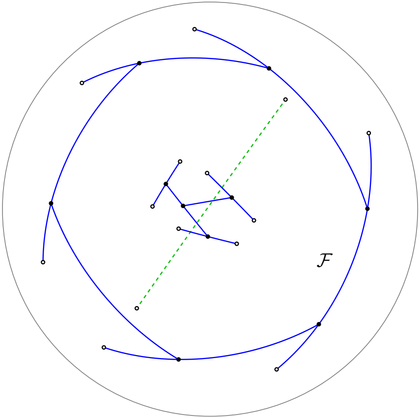


Proof. If there are two connected components, each of them is a Diagram. So, one is contained in a tile $\mathcal{F}$ determined by the other.

Spherical Occlusion Diagrams: basic properties

Corollary

Every Diagram is connected.

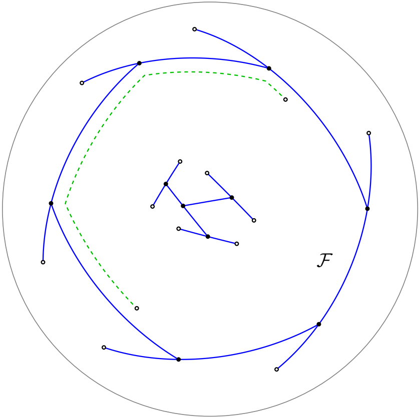


Take an arc in $\mathcal{F}$ with endpoints close to the first component that intersects the second component.

Spherical Occlusion Diagrams: basic properties

Corollary

Every Diagram is connected.

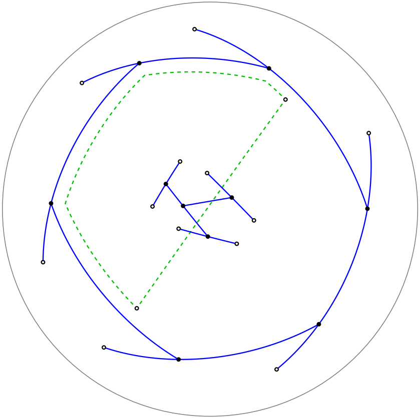


The arc can be replaced by a chain that intersects neither connected component of the Diagram.

Spherical Occlusion Diagrams: basic properties

Corollary

Every Diagram is connected.

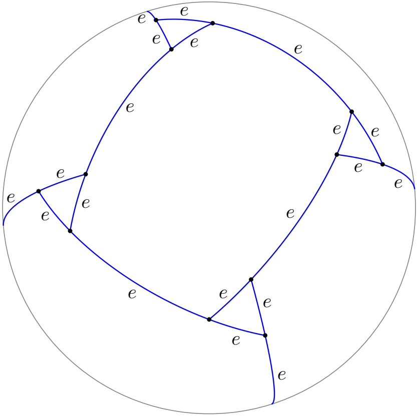


So its endpoints are in the same tile determined by the whole Diagram, and this tile cannot be convex.

Spherical Occlusion Diagrams: basic properties

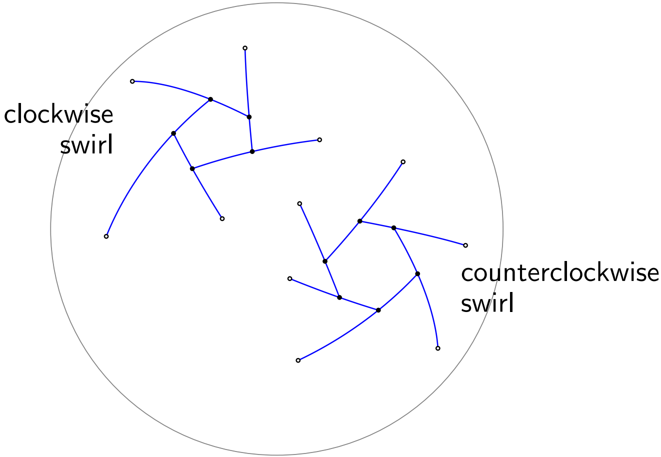
Proposition

A Diagram with n arcs partitions the sphere into $n+2$ tiles.



Proof. A Diagram induces a planar graph with v vertices and $n+v$ edges. By Euler's formula, $f+v=n+v+2$, hence $f=n+2$.

Spherical Occlusion Diagrams: swirls

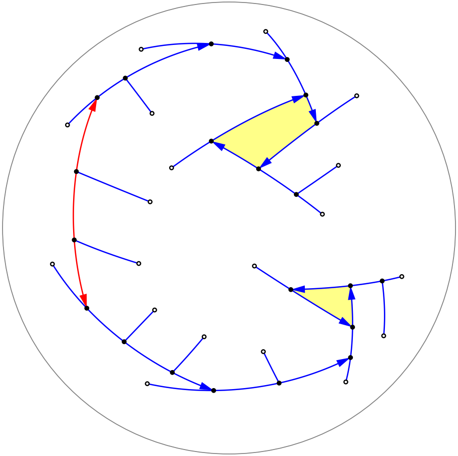


A **swirl** in a Diagram is a cycle of arcs such that each arc feeds into the next going clockwise or counterclockwise.

Spherical Occlusion Diagrams: swirls

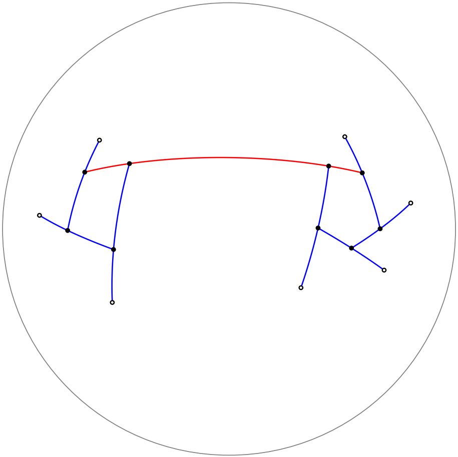
Proposition

Every Diagram contains a clockwise and a counterclockwise swirl.



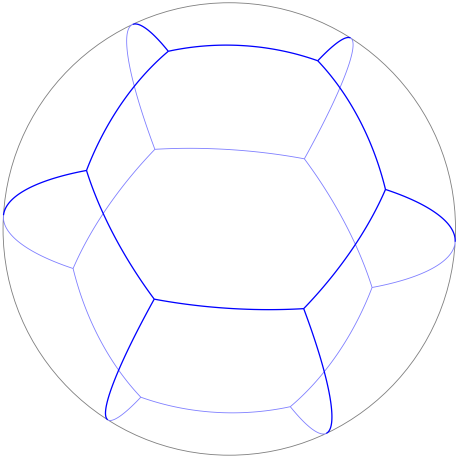
Proof. Start anywhere and follow the Diagram (counter)clockwise.

Spherical Occlusion Diagrams: swirls



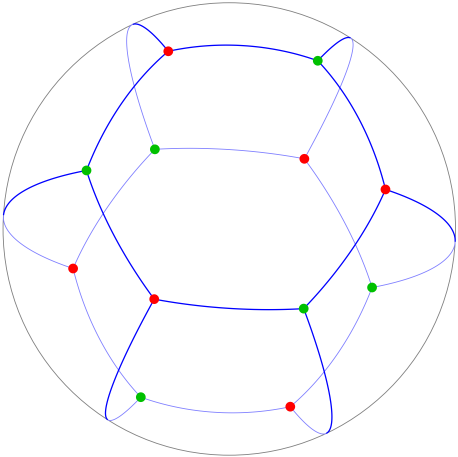
A Diagram is **swirling** if every arc is part of two swirls (note that one swirl must be clockwise and the other counterclockwise).

Spherical Occlusion Diagrams: swirls



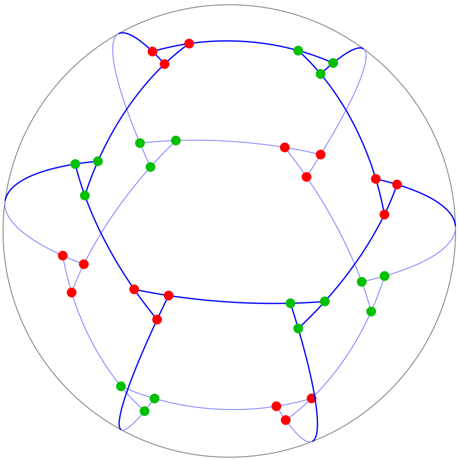
Consider a subdivision of the sphere into strictly convex tiles, where each tile has an even number of edges.

Spherical Occlusion Diagrams: swirls



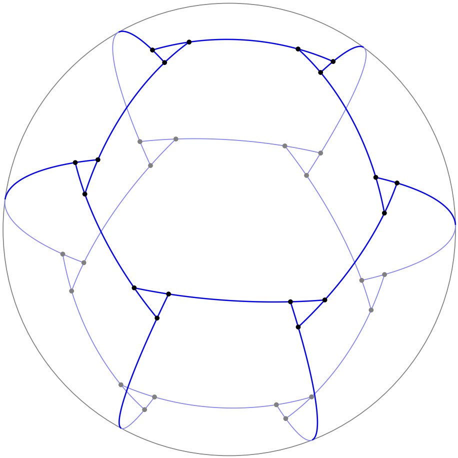
Note that the 1-skeleton of the tiling is bipartite, because it has no odd cycles.

Spherical Occlusion Diagrams: swirls



We can turn each vertex of the tiling into a swirl, going clockwise or counterclockwise according to the bipartition of the 1-skeleton.

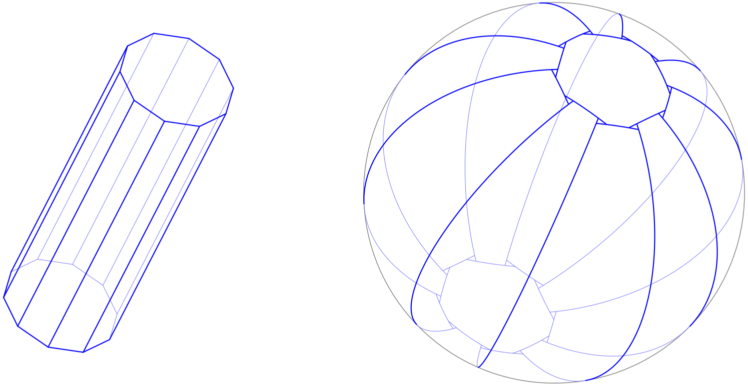
Spherical Occlusion Diagrams: swirls



This operation defines a natural correspondence between swirling Diagrams and even-sided spherical tilings.

Spherical Occlusion Diagrams: swirls

This method enables the automatic construction of swirling Diagrams from convex tilings of the sphere or convex polyhedra.



Prisms with even-sided bases

Spherical Occlusion Diagrams: swirls

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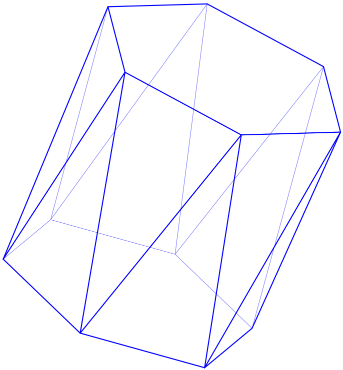

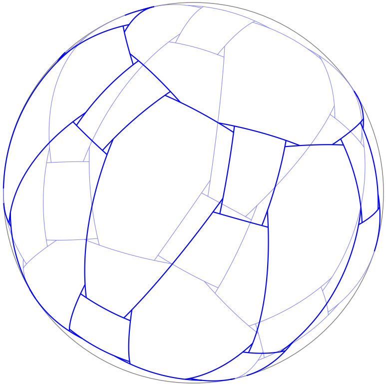

Truncated antiprisms

Spherical Occlusion Diagrams: swirls

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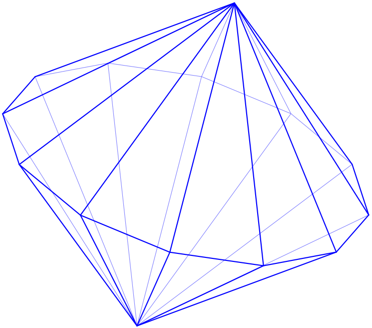

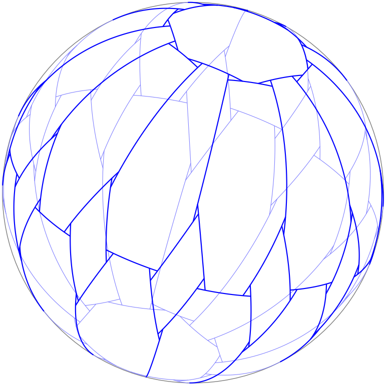

Truncated bipyramids with even-degree vertices

Spherical Occlusion Diagrams: swirls

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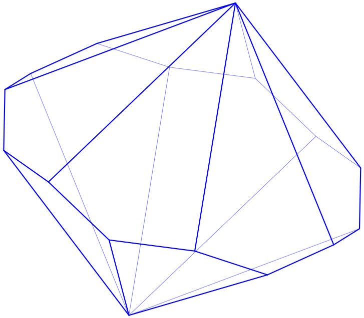
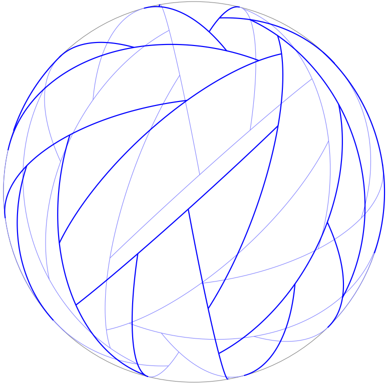
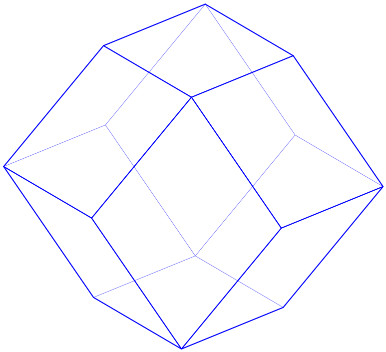


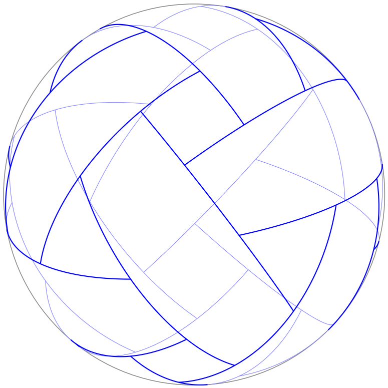


Trapezohedra

Spherical Occlusion Diagrams: swirls

This method enables the <u>automatic construction</u> of swirling Diagrams from convex tilings of the sphere or convex polyhedra.

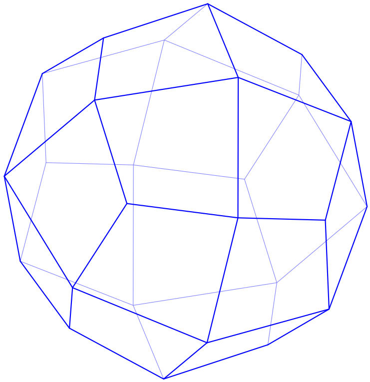


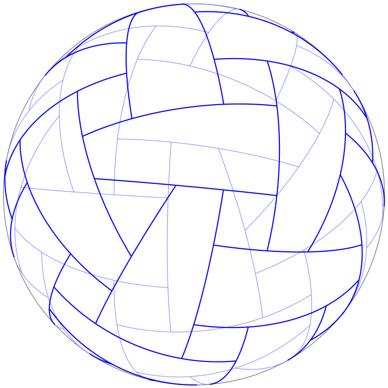


Rhombic dodecahedron

Spherical Occlusion Diagrams: swirls

This method enables the automatic construction of swirling Diagrams from convex tilings of the sphere or convex polyhedra.

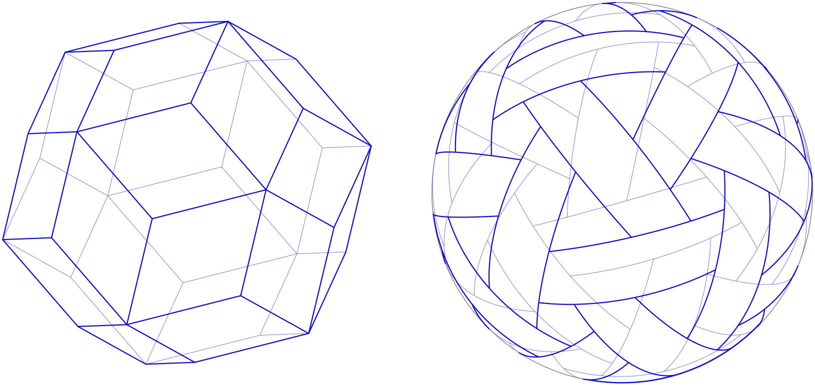




Deltoidal icositetrahedron

Spherical Occlusion Diagrams: swirls

This method enables the automatic construction of swirling Diagrams from convex tilings of the sphere or convex polyhedra.



Rhombic triancontahedron

Spherical Occlusion Diagrams: swirls

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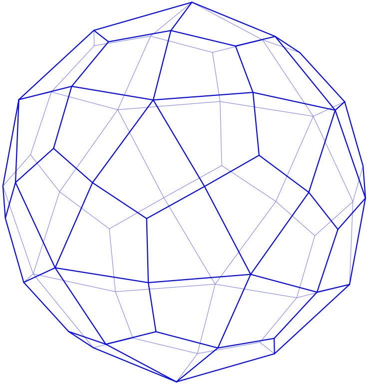

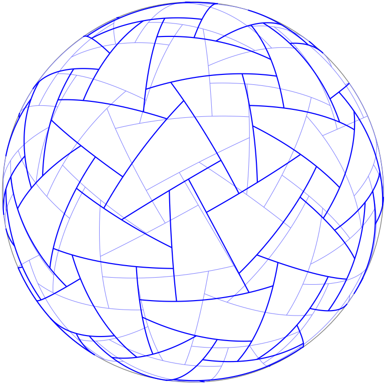

Deltoidal hexecontahedron

Spherical Occlusion Diagrams: swirls

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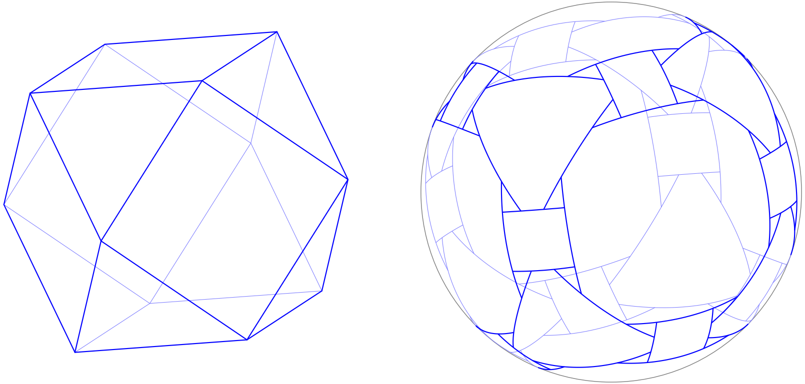
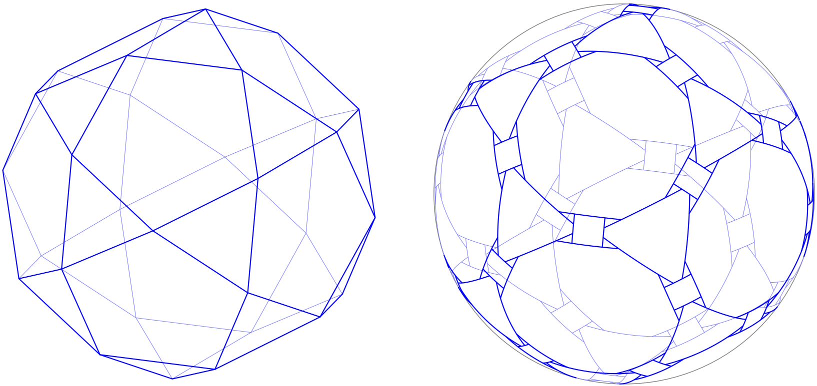


Truncated cuboctahedron

Spherical Occlusion Diagrams: swirls

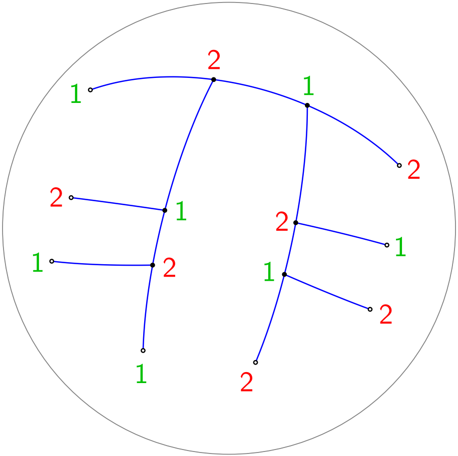
This method enables the automatic construction of swirling Diagrams from convex tilings of the sphere or convex polyhedra.



Truncated icosidodecahedron

Spherical Occlusion Diagrams: uniformity

Each arc in a Diagram feeds into exactly two arcs. So, the average number of arcs feeding into a given arc of a Diagram is two.

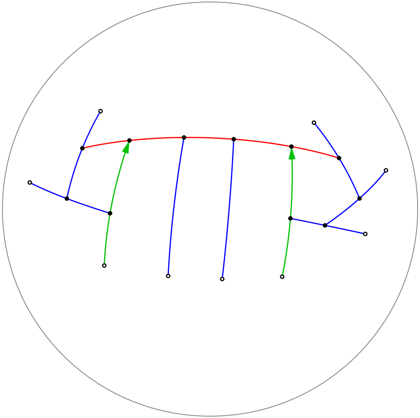


A Diagram is said **uniform** if each arc has two arcs feeding into it.

Spherical Occlusion Diagrams: uniformity

Proposition

All swirling Diagrams are uniform.

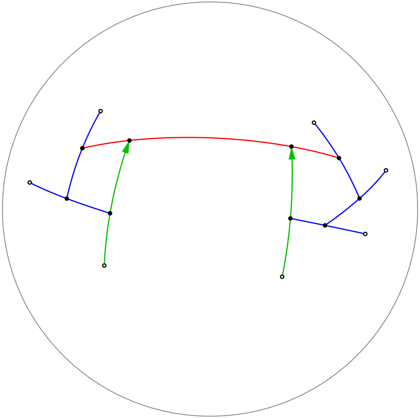


Proof. In a swirling Diagram, each arc is part of two distinct swirls, and so at least two arcs feed into it.

Spherical Occlusion Diagrams: uniformity

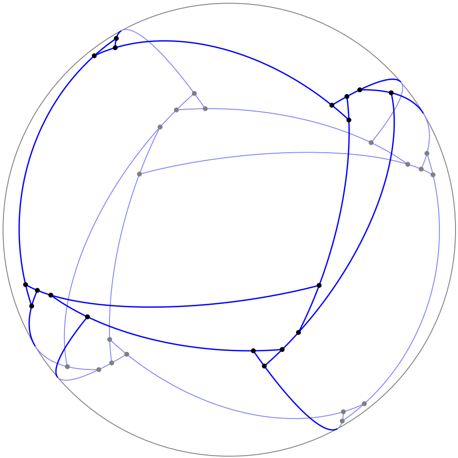
Proposition

All swirling Diagrams are uniform.



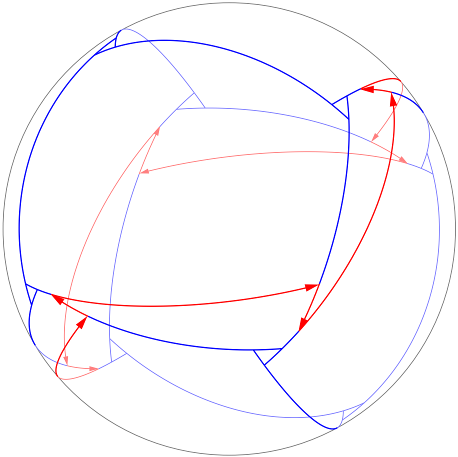
But each arc has two arcs feeding into it on average, so it must have exactly two arcs feeding into it.

Spherical Occlusion Diagrams: uniformity



The converse is not true: there are <u>uniform Diagrams</u> that are <u>not swirling</u>.

Spherical Occlusion Diagrams: uniformity

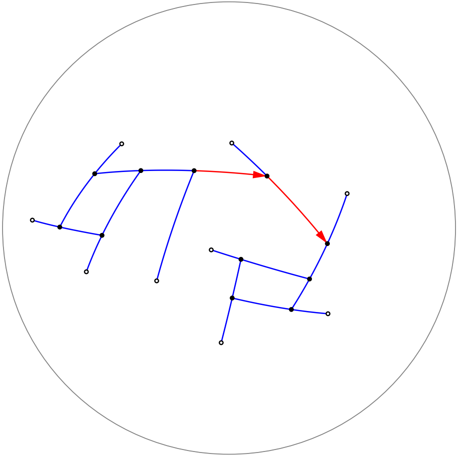


Note that the (portions of) arcs that are not part of a swirl form a cycle where each arc feeds into the next: *this is not a coincidence...*

Spherical Occlusion Diagrams: uniformity

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.

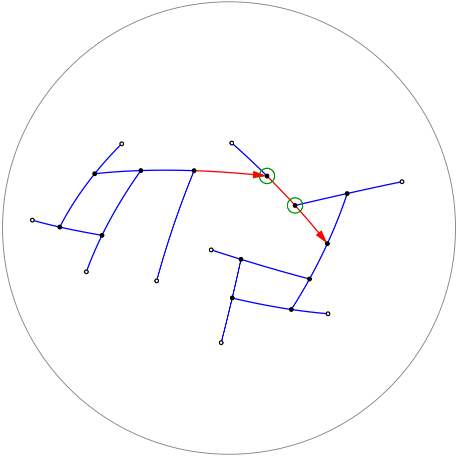


Proof. Consider the last arc in a chain of non-swirling arcs.

Spherical Occlusion Diagrams: uniformity

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.

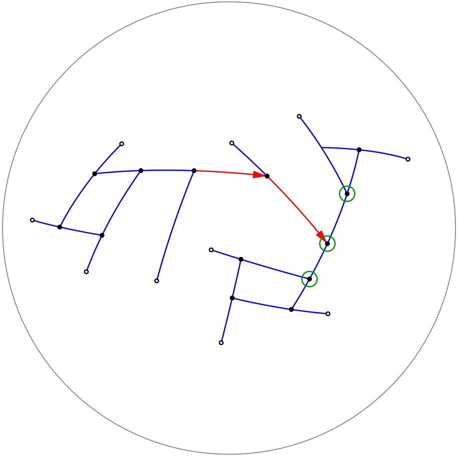


This arc cannot form a swirl with the arc it feeds into (axiom 3).

Spherical Occlusion Diagrams: uniformity

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.

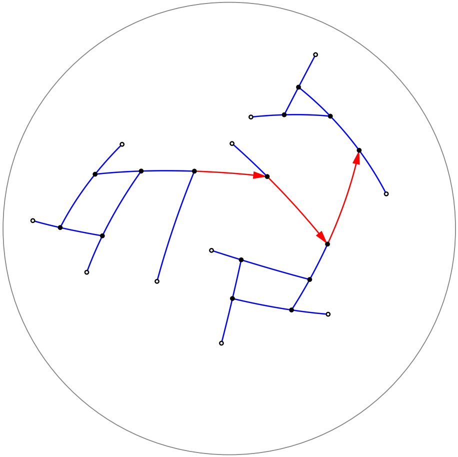


So, the arc it feeds into cannot be part of two swirls (uniformity).

Spherical Occlusion Diagrams: uniformity

Proposition

In a uniform Diagram, the non-swirling arcs form disjoint cycles.



Therefore, the chain must be followed by another non-swirling arc.

Spherical Occlusion Diagrams: uniformity

Proposition

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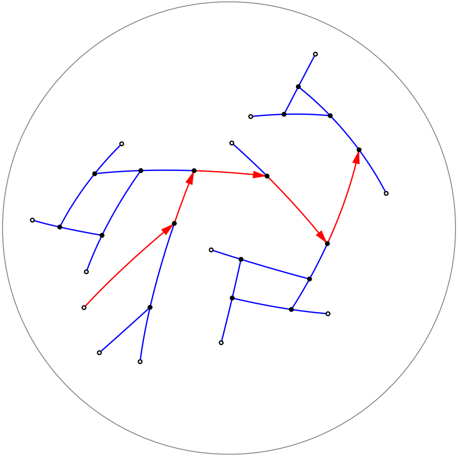
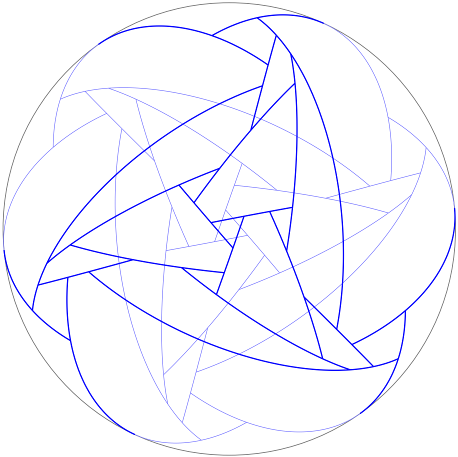
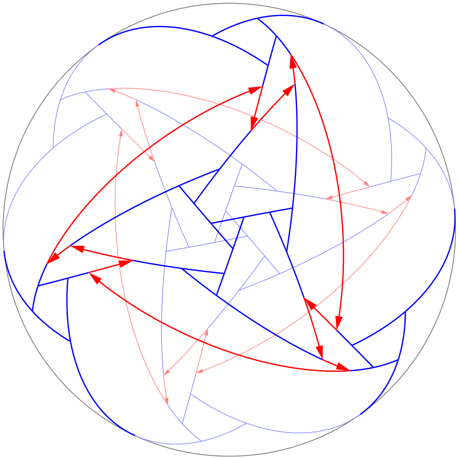


Moreover, the chain can be uniquely extended backwards.

Spherical Occlusion Diagrams: uniformity



Uniform Diagrams can have any number of unboundedly long cycles of non-swirling arcs.



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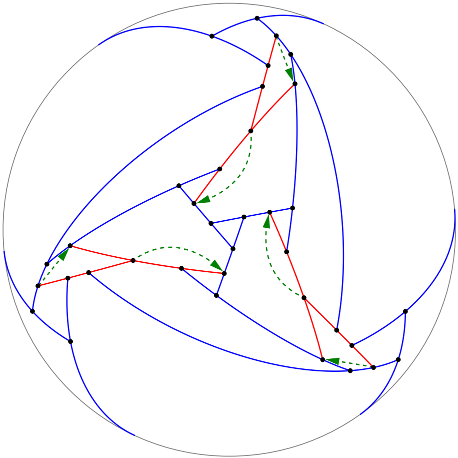
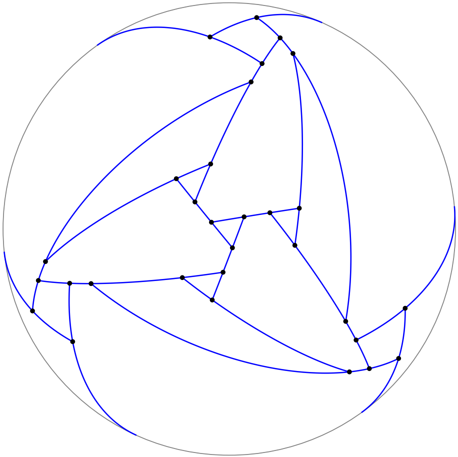
Spherical Occlusion Diagrams: uniformity



By suitably merging consecutive arcs in each cycle, we can transform any uniform Diagram into a swirling one.

Spherical Occlusion Diagrams: uniformity



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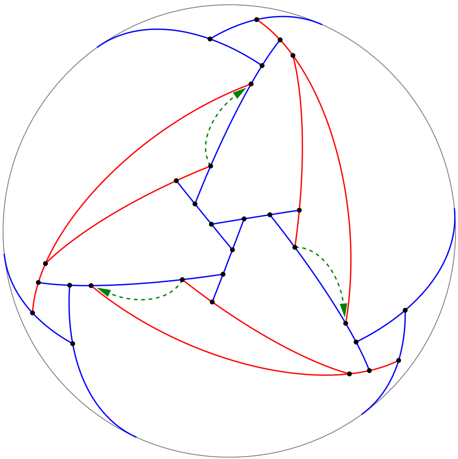
Spherical Occlusion Diagrams: uniformity



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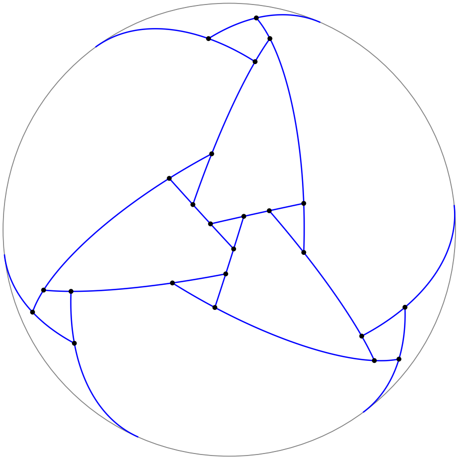
Spherical Occlusion Diagrams: uniformity



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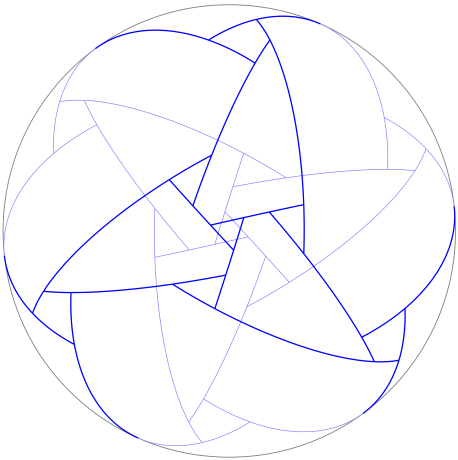
Spherical Occlusion Diagrams: uniformity



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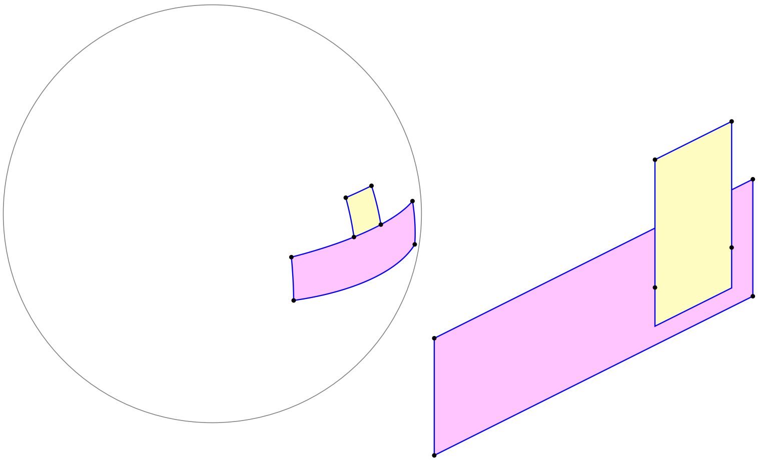
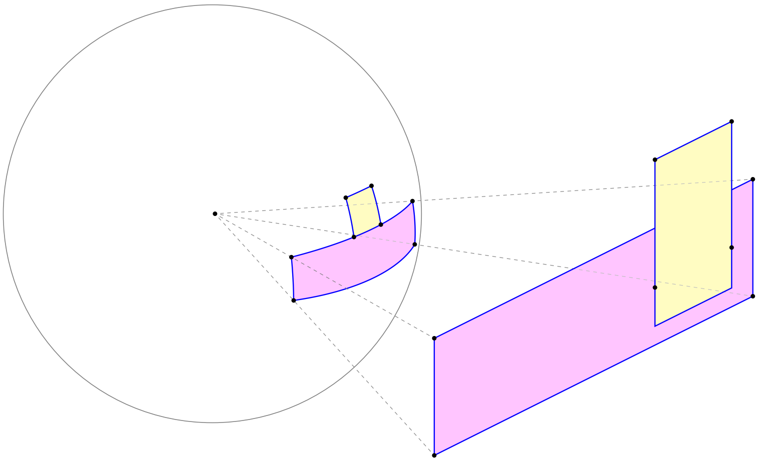
Spherical Occlusion Diagrams: a different perspective



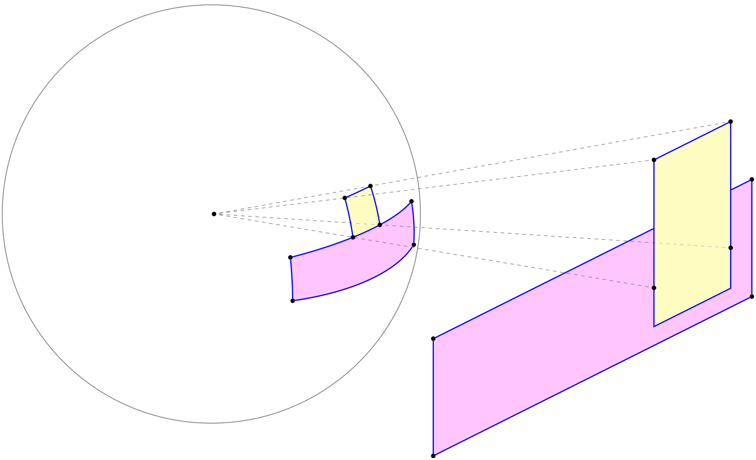
When polygons in $\mathbb{R}^3$ are orthographically projected onto a sphere, their edges become arcs of great circle.

Spherical Occlusion Diagrams: a different perspective



Moreover, when a polygon is partially hidden (i.e., **"occluded"**) by another, in the projection there are arcs feeding into other arcs.

Spherical Occlusion Diagrams: a different perspective



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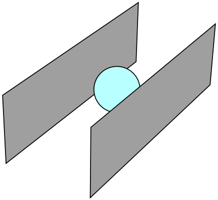
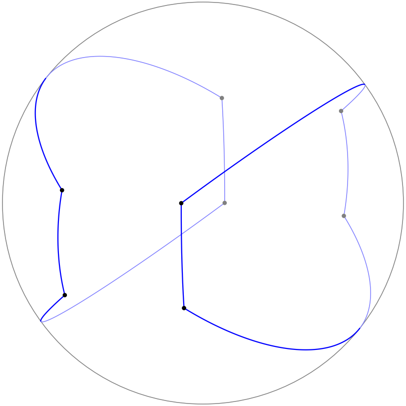
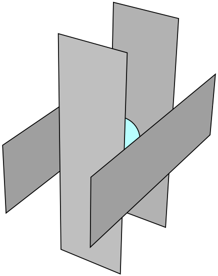
Spherical Occlusion Diagrams: a different perspective

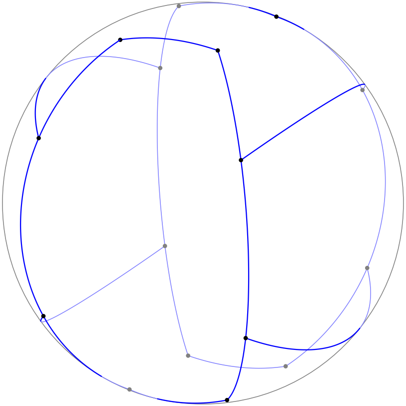




If in an arrangement of polygons all vertices are occluded, then their edges project into a Spherical Occlusion Diagram.

Spherical Occlusion Diagrams: a different perspective





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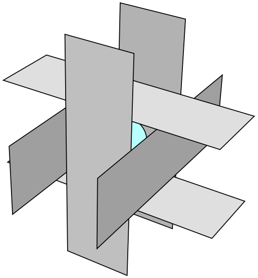
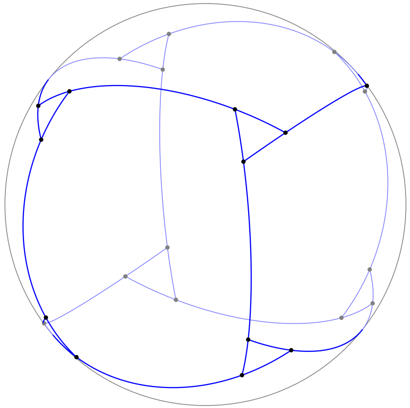
Spherical Occlusion Diagrams: a different perspective





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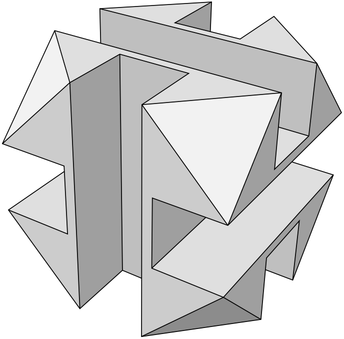
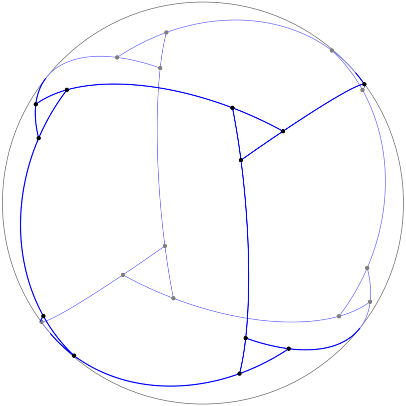
Spherical Occlusion Diagrams: a different perspective



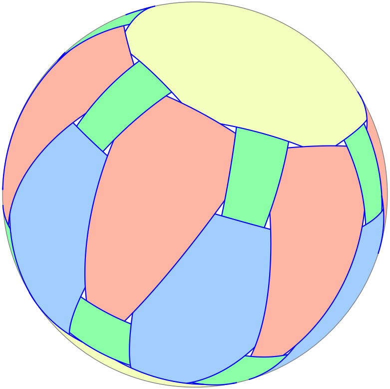


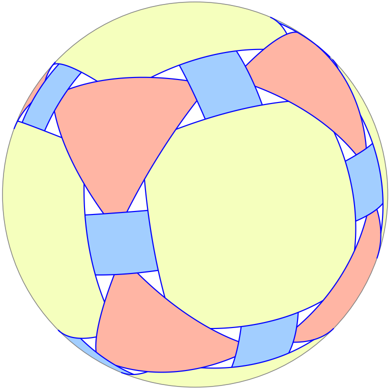
In particular, this applies to polyhedra: if all vertices are occluded, then the 1-skeleton projects into a Spherical Occlusion Diagram.

Spherical Occlusion Diagrams: a different perspective

Observation

If in an arrangement of polygons all vertices are occluded, and each edge occludes vertices of at most one polygon, then the edges project into a swirling Diagram.





Future work

Conjecture

There are no Diagrams with fewer than 12 arcs. There are no swirling Diagrams with 13, 14, 15, 17, 21, 22, 23, or 29 arcs.

Conjecture

Every Diagram is a projection of some polyhedron's 1-skeleton.

Conjecture

Any Diagram can be constructed by a sequence of "elementary operations" starting from a swirling Diagram (e.g., continuously shifting arcs' endpoints or adding arcs).

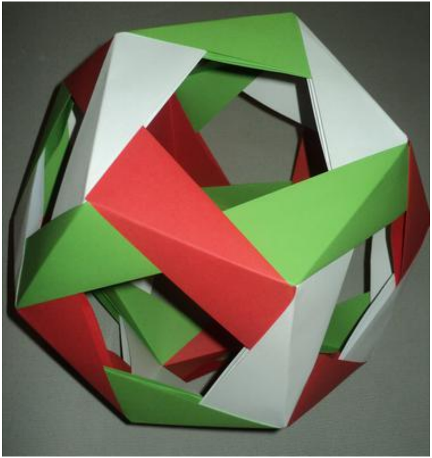
Open problem

Find more contexts where Diagrams naturally arise, and find more applications of the theory of Diagrams.

Diagrams in everyday life

Modular origami: kusudama

Diagrams in everyday life



Modular origami: penultimate dodecahedron

Diagrams in everyday life

Modular origami: penultimate truncated icosahedron

Diagrams in everyday life

Kirigami ball decoration

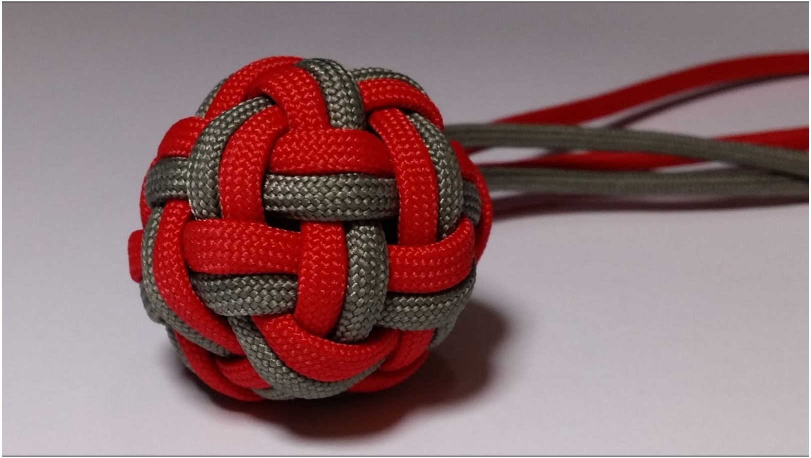
Diagrams in everyday life

Monkey's fist knot

Diagrams in everyday life

Single-thread globe knot

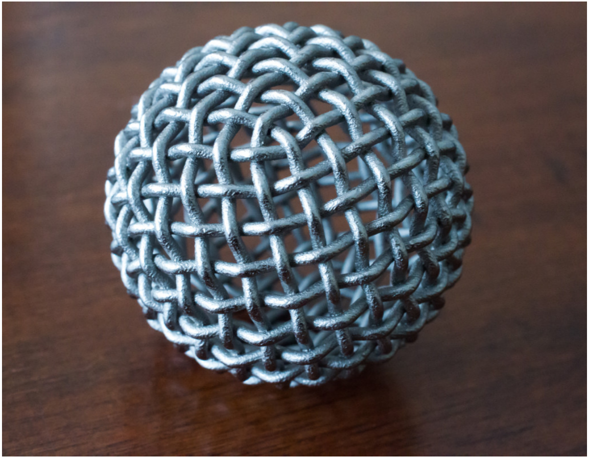
Diagrams in everyday life



Double-thread globe knot

Diagrams in everyday life

Herringbone pineapple knot



Stainless-steel globe knot

Diagrams in everyday life

Sepak-takraw ball

Rattan balls

Diagrams in everyday life

Rattan vase

Diagrams in everyday life

Toroidal Occlusion Diagrams...?