The Bellows Theorem (Proof)

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JAIST – July 5, 2018

Bellows theorem: statement

Theorem (Sabitov, 1996)

The volume V *of a polyhedron (of any genus) with edge lengths* $\ell_1, \cdots, \ell_e$ *satisfies*

$$V^N + A_{N-1}V^{N-1} + \cdots + A_2V^2 + A_1V + A_0 = 0,$$

where the coefficients A_i *are polynomials in* $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$ *and only depend on the combinatorial structure of the polyhedron.*

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where the coefficients A_i are polynomials in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$ and only depend on the combinatorial structure of the polyhedron.

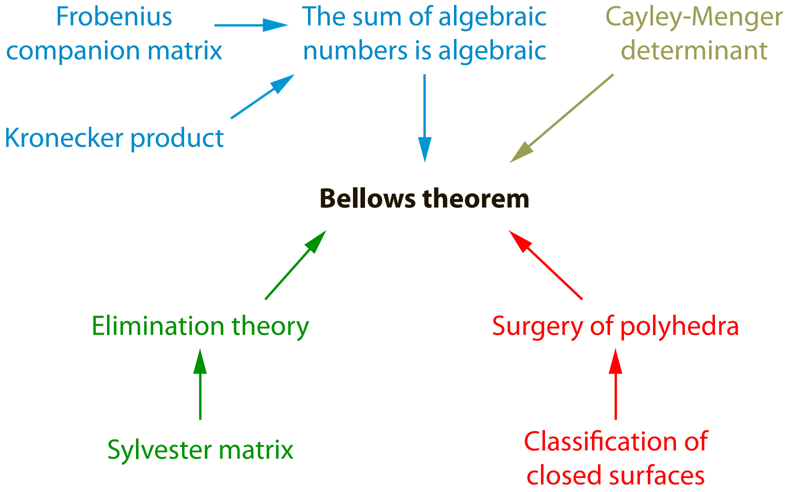
As the polyhedron flexes maintaining its edge lengths ℓ_i fixed, the coefficients A_i remain the same. Hence the volume V is a root of the same polynomial, and it can only take finitely many values.

Corollary (Bellows theorem)

The volume of a polyhedron is constant throughout any flexing.

Note: for the sake of the bellows theorem, it is not restrictive to consider only polyhedra with triangular faces.

Proof roadmap



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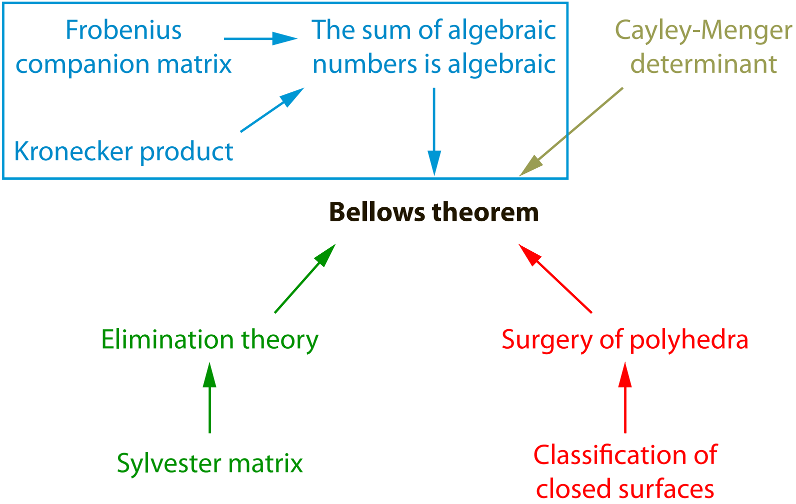

Characteristic polynomial of a matrix

The characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ is the monic degree-n polynomial $c_{\mathbf{A}}(x) = \det(x\mathbf{I} - \mathbf{A})$.

Example:

The characteristic polynomial of the matrix $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ is

$$\det\left(x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\right) = \det \begin{bmatrix} x-2 & -1 \\ 1 & x \end{bmatrix} = x^2 - 2x + 1$$

Lemma

The roots of $c_{\mathbf{A}}(x)$ are precisely the eigenvalues of $\mathbf{A}$.

Frobenius companion matrix of a polynomial

The Frobenius companion matrix of the monic polynomial

$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is the $n \times n$ matrix:

$$\mathcal{F}_P = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

Frobenius companion matrix of a polynomial

Lemma

The eigenvalues of $\mathcal{F}_P$ are precisely the roots of $P(x)$.

Let us prove by induction on n that $c_{\mathcal{F}_P}(x) = P(x)$.

$$c_{\mathcal{F}_P}(x) = \det(x\mathbf{I} - \mathcal{F}_P) = \det \begin{bmatrix} x & 0 & \cdots & 0 & a_0 \\ -1 & x & \cdots & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x + a_{n-1} \end{bmatrix} =$$

$$x \cdot \det \begin{bmatrix} x & 0 & \cdots & 0 & a_1 \\ -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & a_{n-2} \\ 0 & 0 & \cdots & -1 & x + a_{n-1} \end{bmatrix} + (-1)^{n+1} a_0 \cdot \det \begin{bmatrix} -1 & x & 0 & \cdots & 0 \\ 0 & -1 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

$$= x \cdot (x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1) + (-1)^{n+1} a_0 \cdot (-1)^{n-1}$$
$$= x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = P(x) \qquad \square$$

Kronecker product

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $mp \times nq$ matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} & \boxed{2 \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}} \\ 3 \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} & 4 \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{bmatrix}$$

Kronecker product: mixed-product property

If the matrix products $\mathbf{AC}$ and $\mathbf{BD}$ are well defined, then:

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix} \begin{bmatrix} c_{11}\mathbf{D} & \dots & c_{1p}\mathbf{D} \\ \vdots & \ddots & \vdots \\ c_{n1}\mathbf{D} & \dots & c_{np}\mathbf{D} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}c_{k1}\mathbf{BD} & \dots & \sum_{k=1}^{n} a_{1k}c_{kp}\mathbf{BD} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}c_{k1}\mathbf{BD} & \dots & \sum_{k=1}^{n} a_{mk}c_{kp}\mathbf{BD} \end{bmatrix} = \mathbf{AC} \otimes \mathbf{BD}$$

The sum of polynomial roots is a polynomial root

Lemma

If A and B are monic polynomials with coefficients in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$, there is a monic polynomial C with coefficients in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$ such that, if $A(\alpha) = 0$ and $B(\beta) = 0$, then $C(\alpha + \beta) = 0$.

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Let $\mathbf{A}$ and $\mathbf{B}$ be the Frobenius companion matrices of A and B.

Then there are vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{Ax} = \alpha\mathbf{x}$ and $\mathbf{By} = \beta\mathbf{y}$.

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$$(\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B})(\mathbf{x} \otimes \mathbf{y}) = (\mathbf{A} \otimes \mathbf{I})(\mathbf{x} \otimes \mathbf{y}) + (\mathbf{I} \otimes \mathbf{B})(\mathbf{x} \otimes \mathbf{y})$$

(apply the mixed-product property)

$$= (\mathbf{A}\mathbf{x} \otimes \mathbf{I}\mathbf{y}) + (\mathbf{I}\mathbf{x} \otimes \mathbf{B}\mathbf{y}) = (\alpha\mathbf{x} \otimes \mathbf{y}) + (\mathbf{x} \otimes \beta\mathbf{y}) = (\alpha + \beta)(\mathbf{x} \otimes \mathbf{y})$$

Hence $\alpha + \beta$ is an eigenvalue of the matrix $\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B}$, and therefore $\alpha + \beta$ is a root of its characteristic polynomial C.

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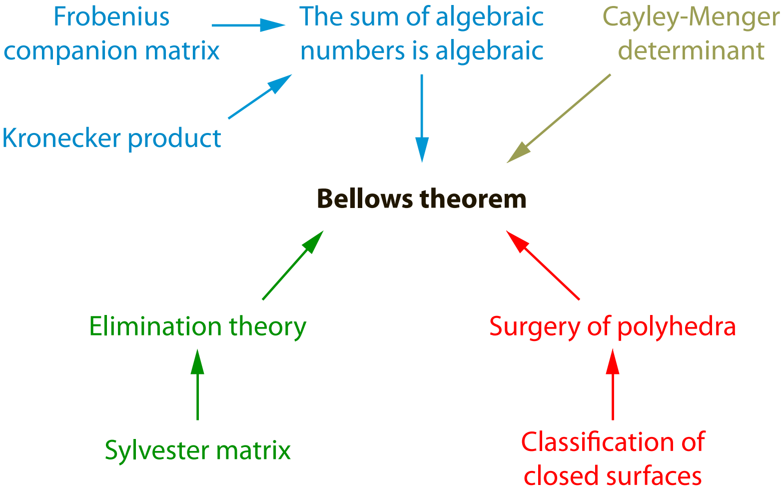
Hence $\alpha + \beta$ is an eigenvalue of the matrix $\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B}$, and therefore $\alpha + \beta$ is a root of its characteristic polynomial C.

The coefficients of C were obtained by adding and multiplying coefficients of A and B, and thus they are in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$. $\square$

Proof roadmap



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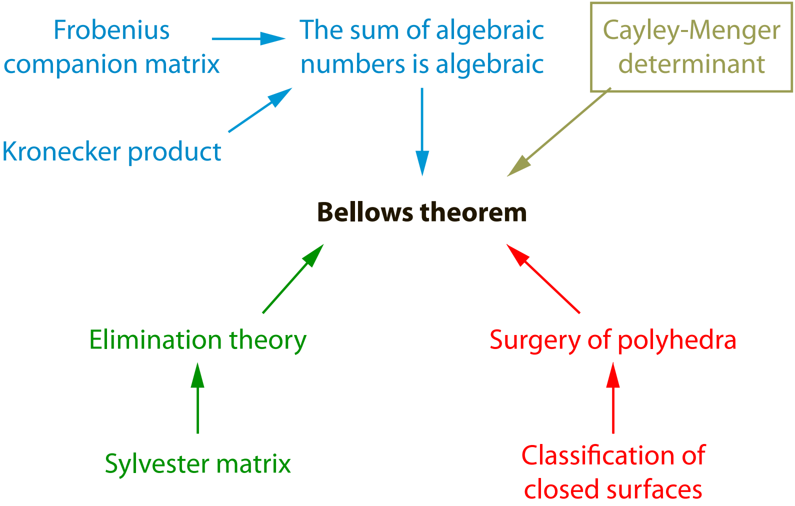

Cayley-Menger determinant

Lemma

If $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^3$ *and* $d_{ij} = \|x_i - x_j\|$, *then*

$$\det \begin{bmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 & d_{25}^2 & 1 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & d_{35}^2 & 1 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 & d_{45}^2 & 1 \\ d_{51}^2 & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} = 0.$$

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The matrix is the product of these two:

$$\mathbf{A} = \begin{bmatrix} \|x_1\|^2 & -2x_{11} & -2x_{12} & -2x_{13} & 1 & 0 \\ \|x_2\|^2 & -2x_{21} & -2x_{22} & -2x_{23} & 1 & 0 \\ \|x_3\|^2 & -2x_{31} & -2x_{32} & -2x_{33} & 1 & 0 \\ \|x_4\|^2 & -2x_{41} & -2x_{42} & -2x_{43} & 1 & 0 \\ \|x_5\|^2 & -2x_{51} & -2x_{52} & -2x_{53} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ x_{11} & x_{21} & x_{31} & x_{41} & x_{51} & 0 \\ x_{12} & x_{22} & x_{32} & x_{42} & x_{52} & 0 \\ x_{13} & x_{23} & x_{33} & x_{43} & x_{53} & 0 \\ \|x_1\|^2 & \|x_2\|^2 & \|x_3\|^2 & \|x_4\|^2 & \|x_5\|^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e.g., $\|x_1\|^2 - 2x_{11}^2 - 2x_{12}^2 - 2x_{13}^2 + \|x_1\|^2 = \|x_1 - x_1\|^2 = 0$

$\|x_3\|^2 - 2x_{31}x_{41} - 2x_{32}x_{42} - 2x_{33}x_{43} + \|x_4\|^2 = \|x_3 - x_4\|^2 = d_{34}^2$

$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = 0$ ($\mathbf{A}$ has an all-0 column) $\square$

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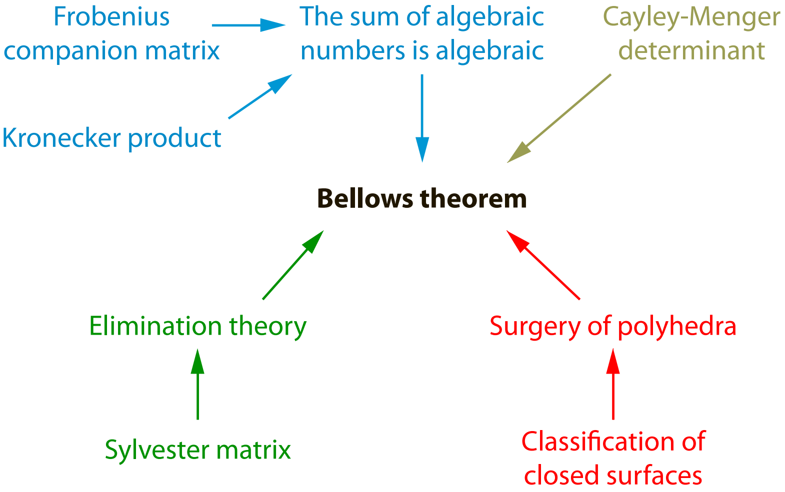

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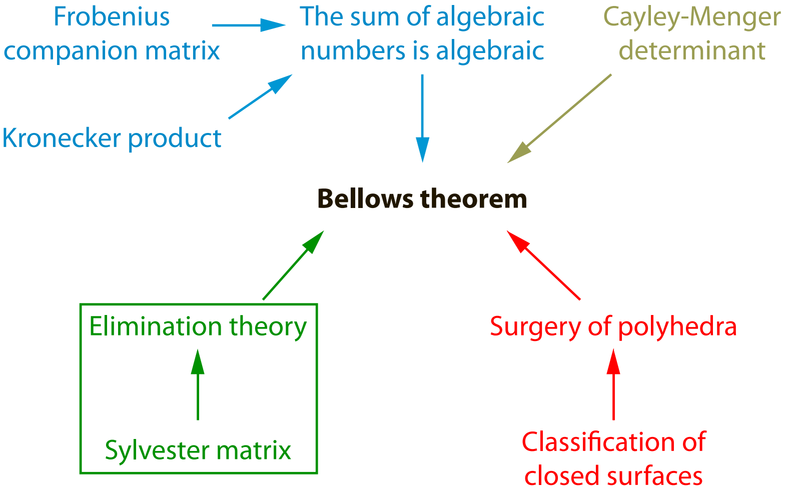

Sylvester matrix

Given two polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$, their <u>Sylvester matrix</u> is:

$$\mathrm{Syl}_{P,Q} = \begin{bmatrix} a_n & & & b_m & & \\ & \ddots & & & \ddots & \\ \vdots & & a_n & \vdots & & \\ & & & b_0 & & b_m \\ a_0 & & \vdots & & & \vdots \\ & \ddots & & & \ddots & \\ & & a_0 & & & b_0 \end{bmatrix}$$

Example: $\deg(P) = 4$, $\deg(Q) = 3$

$$\mathrm{Syl}_{P,Q} = \begin{bmatrix} a_4 & 0 & 0 & b_3 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & b_2 & b_3 & 0 & 0 \\ a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & 0 \\ a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 \\ a_0 & a_1 & a_2 & 0 & b_0 & b_1 & b_2 \\ 0 & a_0 & a_1 & 0 & 0 & b_0 & b_1 \\ 0 & 0 & a_0 & 0 & 0 & 0 & b_0 \end{bmatrix}$$

Elimination theory: single variable

Do two polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ have common roots?

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If so, they have a common non-constant factor $F(x)$:

$P(x) = R(x) \cdot F(x)$ with $\deg(R) < n$

$Q(x) = -S(x) \cdot F(x)$ with $\deg(S) < m$

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$$\left\{ \begin{array}{rcl} a_n s_{m-1} + b_m r_{n-1} & = & 0 \\ a_n s_{m-2} + a_{n-1} s_{m-1} + b_m r_{n-2} + b_{m-1} r_{n-1} & = & 0 \\ \vdots & = & 0 \\ a_0 s_0 + b_0 r_0 & = & 0 \end{array} \right.$$

Elimination theory: single variable

In matrix form,

$$\begin{bmatrix} a_n & & & b_m & & \\ & \ddots & & & \ddots & \\ \vdots & & a_n & \vdots & & \\ & & & b_0 & & b_m \\ a_0 & & \vdots & & & \vdots \\ & \ddots & & & \ddots & \\ & & a_0 & & & b_0 \end{bmatrix} \cdot \begin{bmatrix} s_{m-1} \\ \vdots \\ s_0 \\ r_{n-1} \\ \vdots \\ r_0 \end{bmatrix} = 0$$

with unknowns $s_{m-1}, \cdots s_0, r_{n-1}, \cdots, r_0$.

Equivalently, $\mathrm{Syl}_{P,Q} \cdot \mathbf{x} = 0$, where $\mathbf{x}$ is a non-zero vector.

Equivalently, $\underline{\det(\mathrm{Syl}_{P,Q}) = 0}$.

Elimination theory: multiple variables

Example:

$$\begin{cases} 9x^2 + 4y^2 - 18x + 16y - 11 & = & 0 \\ x^2 + y^2 - 9 & = & 0 \end{cases}$$

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View these as polynomials in y with coefficients polynomials in x.

$$\begin{cases} 4 \cdot y^2 + 16 \cdot y + 9x^2 - 18x - 11 & = & 0 \\ 1 \cdot y^2 + 0 \cdot y + x^2 - 9 & = & 0 \end{cases}$$

This system is solvable if and only if

$$\det \begin{bmatrix} 4 & 0 & 1 & 0 \\ 16 & 4 & 0 & 1 \\ 9x^2 - 18x - 11 & 16 & x^2 - 9 & 0 \\ 0 & 9x^2 - 18x - 11 & 0 & x^2 - 9 \end{bmatrix} = 0$$

Which reduces to the single-variable polynomial equation

$$25x^4 - 180x^3 + 574x^2 - 900x + 625 = 0$$

Proof roadmap

Frobenius companion matrix

The sum of algebraic numbers is algebraic

Cayley-Menger determinant

Kronecker product

Bellows theorem

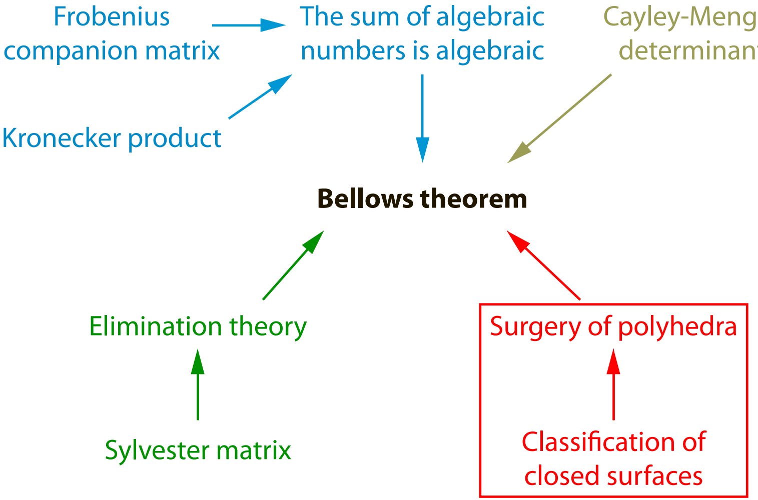
Elimination theory

Surgery of polyhedra

Sylvester matrix

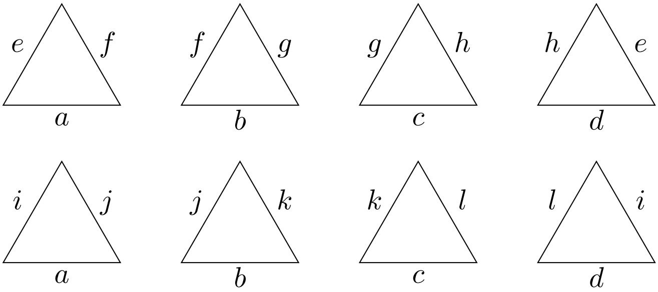
Classification of closed surfaces

Proof roadmap



Combinatorial structure of a polyhedron

Abstract polyhedron:
a set of triangular faces with a perfect matching between edges.



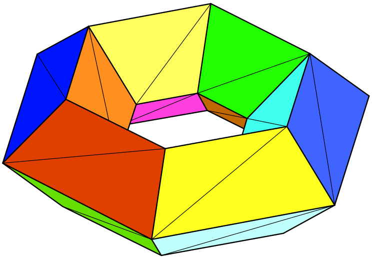
Topologically, this is a closed orientable 2-manifold with V vertices, E edges, F faces, where the Euler-Poincaré formula holds:

$$V - E + F = 2 - 2g,$$

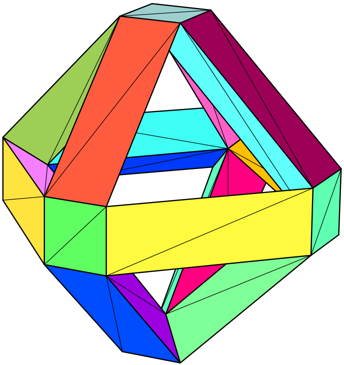
where g is the genus of the polyhedron.

Genus of a polyhedron

The genus of a polyhedron can be visualized as the number of handles on its surface.



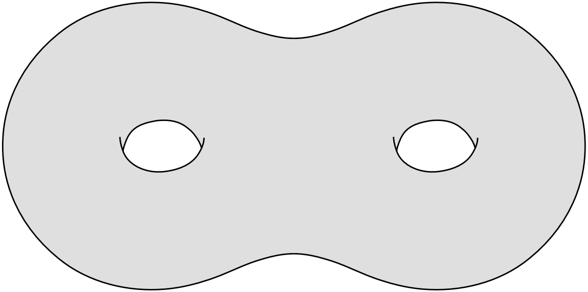
genus = 1



genus = 7

Surgery of closed orientable 2-manifolds

Suppose that a circular cut is made on the surface of a closed orientable 2-manifold, and the cut is patched with two disks.



The result is either one object with strictly lower genus or two objects with equal or lower genus.

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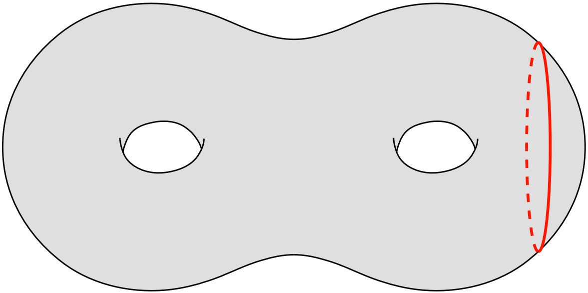

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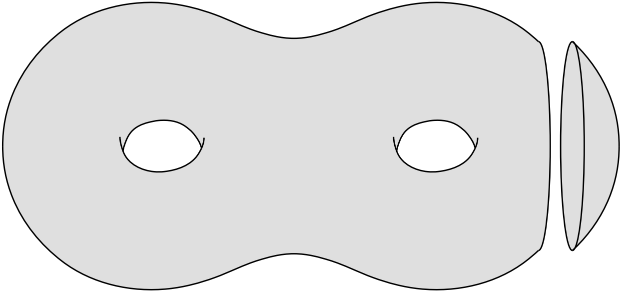

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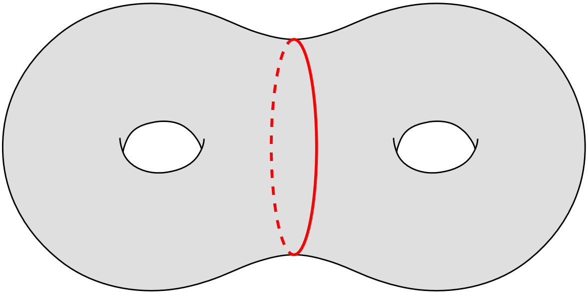

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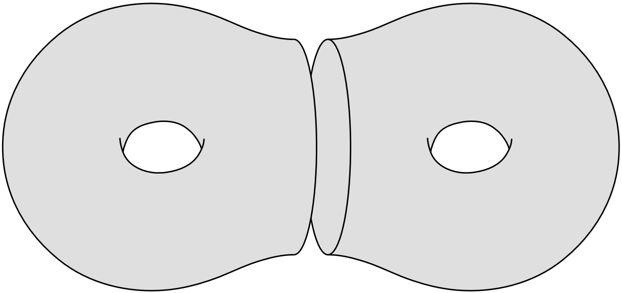

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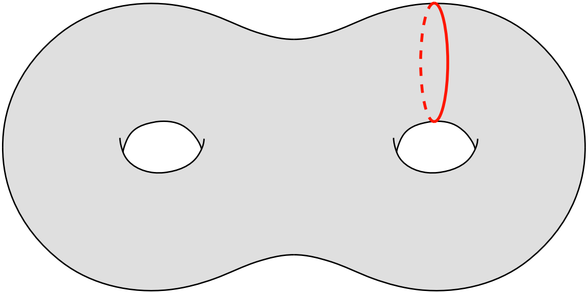

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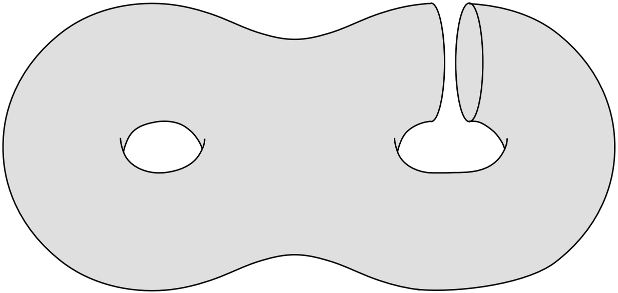

The result is either one object with strictly lower genus or two objects with equal or lower genus.

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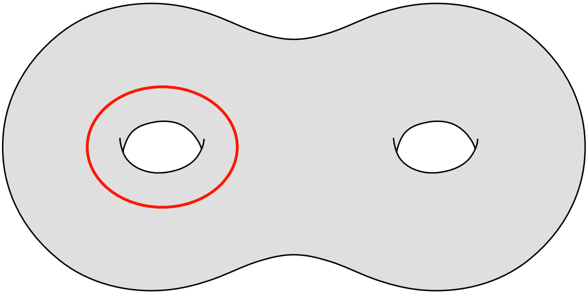

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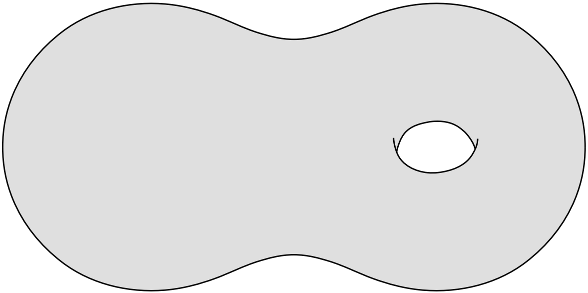

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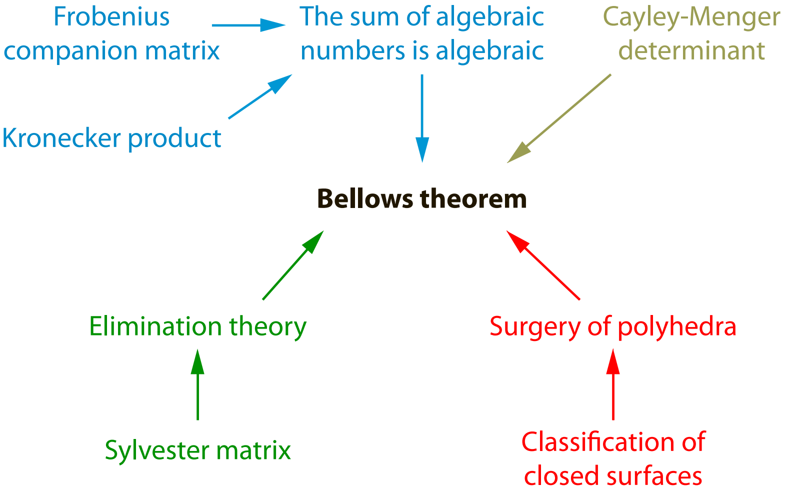
Proof roadmap



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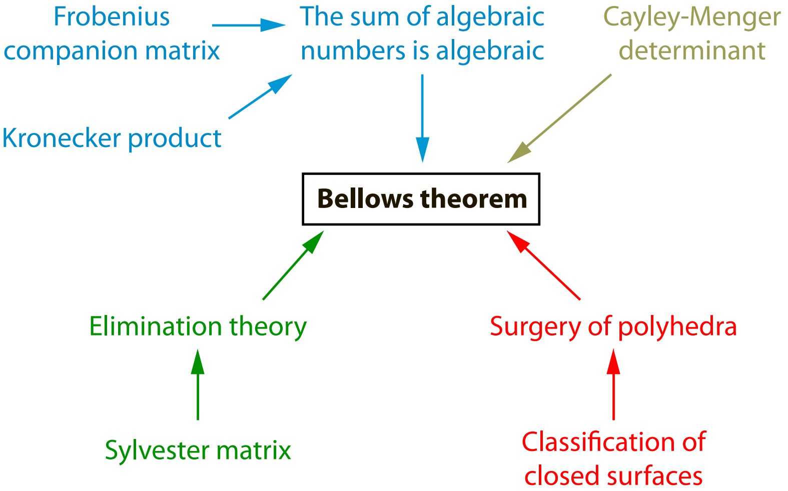
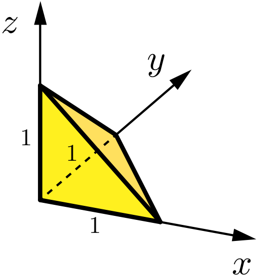
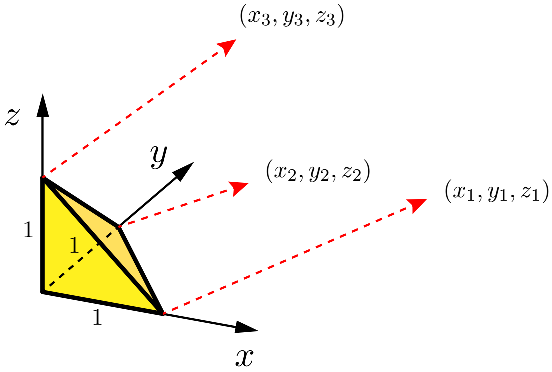


Volume of a tetrahedron



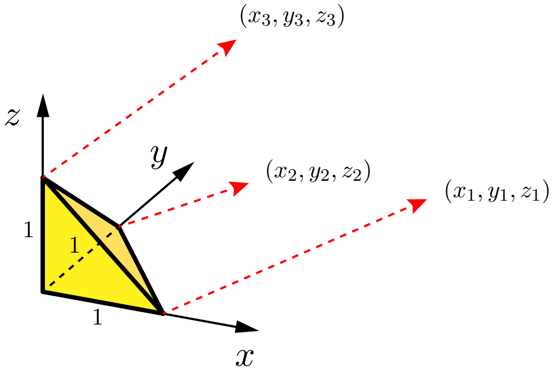
The volume of the tetrahedron is $1/6$.

Volume of a tetrahedron



A linear map sends three vertices to three arbitrary points in $\mathbb{R}^3$.

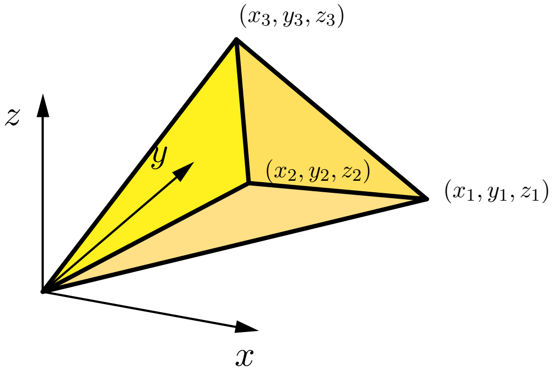
Volume of a tetrahedron



In matrix form, this transformation is

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

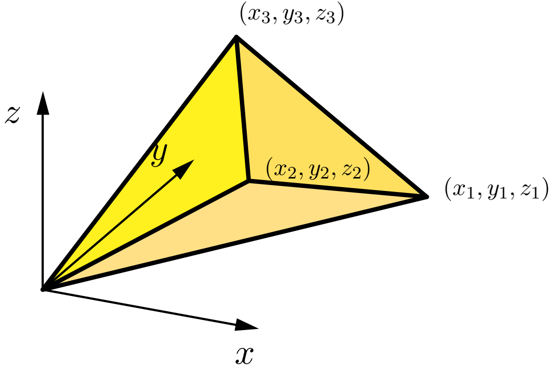
Volume of a tetrahedron



The (signed) volume of the transformed tetrahedron is the volume of the initial tetrahedron multiplied by

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

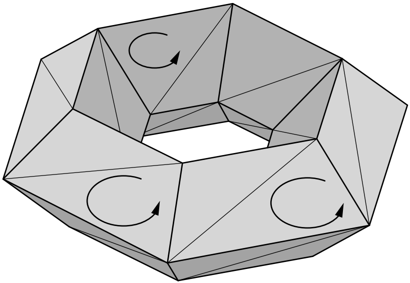
Volume of a tetrahedron



So, the (signed) volume of the new tetrahedron is the polynomial

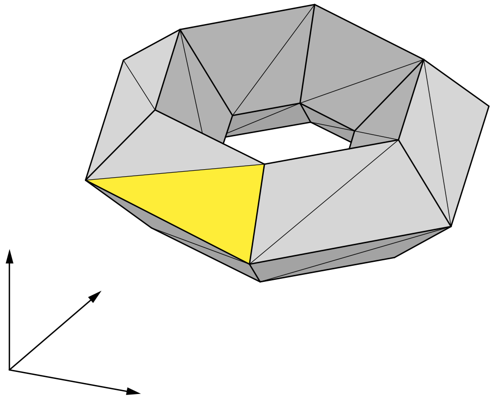
$$\frac{1}{6}\,(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1)$$

Volume polynomial of a polyhedron



Consider a polyhedron, and assign a consistent orientation to each of its faces: e.g., the vertices on a face are taken counterclockwise.

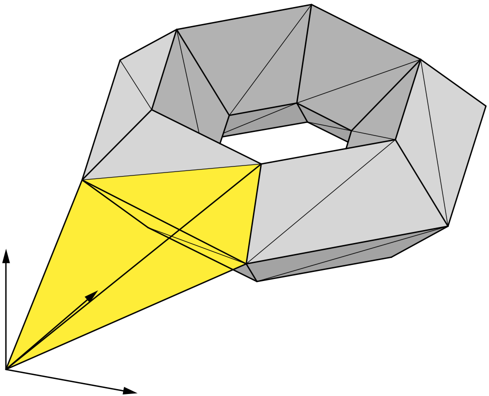
Volume polynomial of a polyhedron



The volume of the polyhedron is the sum of the signed volumes of the tetrahedra spanned by the origin and each face.

Front faces give a positive contribution, and back faces give a negative contribution to the volume.

Volume polynomial of a polyhedron



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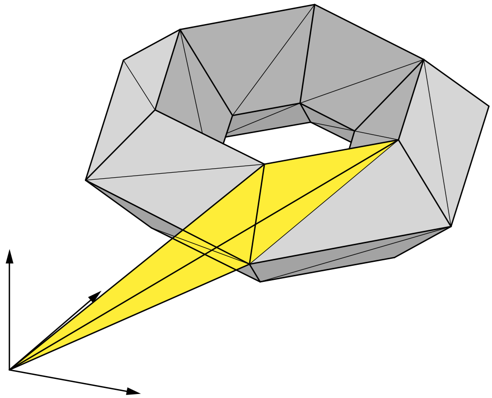

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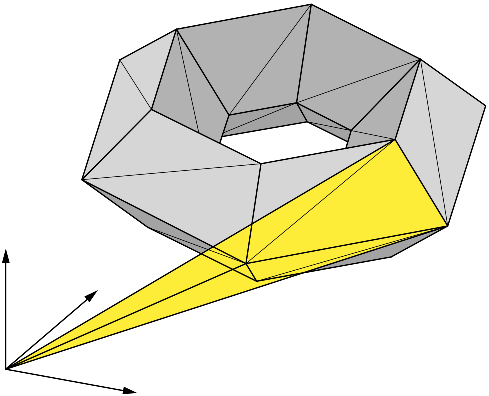

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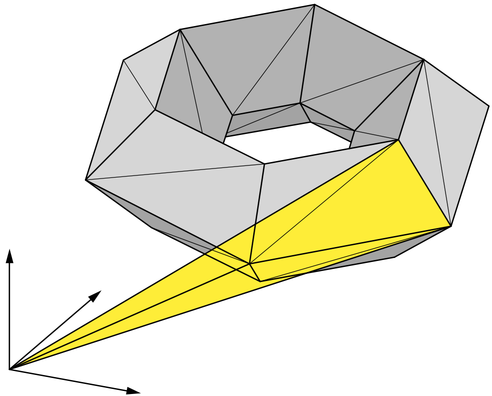
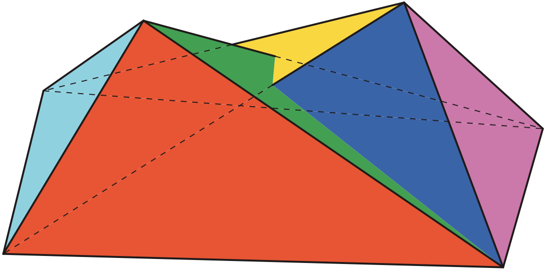


If the coordinates of the n vertices are unknowns, the volume is a <u>polynomial</u> in $\mathbb{Q}[x_1, y_1, z_1, \cdots, x_n, y_n, z_n]$.

Degenerate polyhedra

The definition of volume polynomial also applies to generalized polyhedra with degenerate or intersecting faces, such as the Bricard octahedra.



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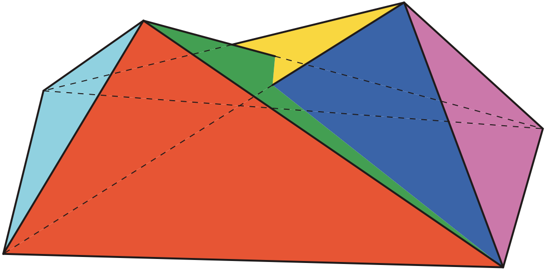

Note: we will need to include these degenerate polyhedra in our theorem, because they may appear as a result of performing surgery on non-degenerate polyhedra.

Bellows theorem: re-statement

Theorem (Sabitov, 1996)

Given the combinatorial structure of a polyhedron, its volume polynomial $V \in \mathbb{Q}[x_1, y_1, z_1, \cdots, x_n, y_n, z_n]$ *satisfies the identity*

$$V^N + A_{N-1}V^{N-1} + \cdots + A_2V^2 + A_1V + A_0 \equiv 0,$$

where $A_i \in \mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$ *and* $\ell_k^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$ *for every edge* $\{(x_i, y_i, z_i), (x_j, y_j, z_j)\}$ *of the polyhedron.*

The theorem expresses an algebraic identity among the unknowns $x_1, y_1, z_1, \cdots, x_n, y_n, z_n$ that is satisfied algebraically after the substitutions $\ell_k^2 := (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$ are made.

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Once we assign values to the edge lengths ℓ_i, the coefficients A_i become numbers, and the polynomial is fixed.

When we also assign coordinates (x_i, y_i, z_i) to the vertices (matching the edge lengths ℓ_i), then also the volume V becomes a number, which must be a root of the polynomial.

Bellows theorem: proof structure

We prove the theorem by induction on some parameters of the combinatorial structure of the polyhedron $\mathcal{P}$, in this order:

- the genus
- the total number of vertices
- the degree of a specific vertex

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- the total number of vertices
- the degree of a specific vertex

The base case is when $\mathcal{P}$ is a tetrahedron.

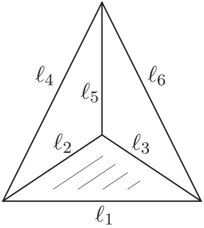
In general, we perform surgery to reduce the complexity of $\mathcal{P}$.

If surgery is not possible, we perform ad-hoc transformations around a vertex and apply the Cayley-Menger determinant to obtain equations which are then simplified using elimination theory.

Base case: tetrahedron

For a tetrahedron, the polynomial equation is

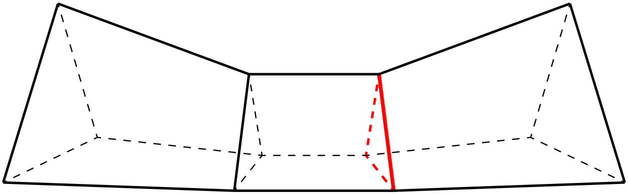
$$V^2 = \frac{1}{144}\left[\ell_1^2\ell_5^2(\ell_2^2+\ell_3^2+\ell_4^2+\ell_6^2-\ell_1^2-\ell_5^2) + \ell_2^2\ell_6^2(\ell_1^2+\ell_3^2+\ell_4^2+\ell_5^2-\ell_2^2-\ell_6^2)\right.$$
$$\left. +\ell_3^2\ell_4^2(\ell_1^2+\ell_2^2+\ell_5^2+\ell_6^2-\ell_3^2-\ell_4^2) - \ell_1^2\ell_2^2\ell_3^2 - \ell_2^2\ell_4^2\ell_5^2 - \ell_1^2\ell_4^2\ell_6^2 - \ell_3^2\ell_5^2\ell_6^2\right]$$



After substituting the volume polynomial for V and $\ell_k^2 := (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$, one can check that all similar monomials cancel out, i.e., this is an algebraic identity.

Empty 3-cycles

If a cycle formed by 3 edges bounds no face, it is called empty.



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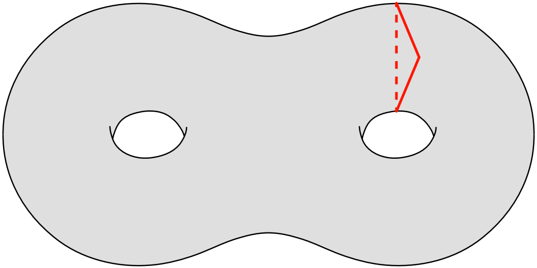

If the polyhedron $\mathcal{P}$ an empty 3-cycle, we perform surgery on it.

If the result is a single polyhedron $\mathcal{P}'$, it must have smaller genus than $\mathcal{P}$, and so the inductive hypothesis applies to $\mathcal{P}'$.

But $\mathcal{P}$ and $\mathcal{P}'$ have the same volume polynomial and the same set of edges. Hence the theorem is true for $\mathcal{P}$.

Empty 3-cycles

If the result of the surgery are two polyhedra $\mathcal{P}'$ and $\mathcal{P}''$, they have equal or smaller genus than $\mathcal{P}$ and strictly fewer vertices.

So the inductive hypothesis holds for $\mathcal{P}'$ and $\mathcal{P}''$.

Note that all the edges of $\mathcal{P}'$ and $\mathcal{P}''$ are also edges of $\mathcal{P}$.

Also, the volume polynomial of $\mathcal{P}$ is the sum of the volume polynomials of $\mathcal{P}'$ and $\mathcal{P}''$.

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Recall:

Lemma

If A and B are monic polynomials with coefficients in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$, there is a monic polynomial C with coefficients in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2]$ such that, if $A(\alpha) = 0$ and $B(\beta) = 0$, then $C(\alpha + \beta) = 0$.

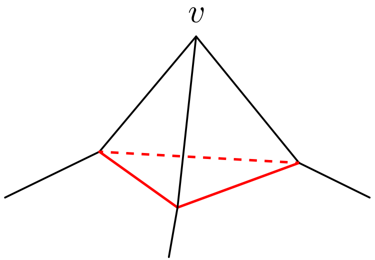
Hence, if A and B are the polynomials for $\mathcal{P}'$ and $\mathcal{P}''$, then C is the polynomial for $\mathcal{P}$.

No empty 3-cycles

Suppose there are no empty 3-cycles, and pick any vertex v.

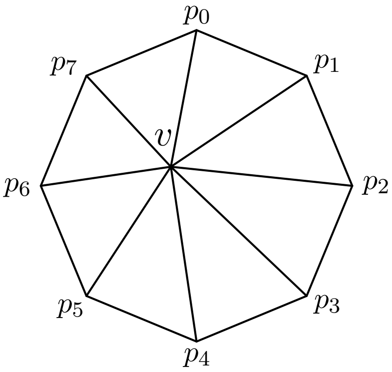
We proceed by induction on the degree of v.

If v has degree 3, then there is an empty 3-cycle: contradiction.



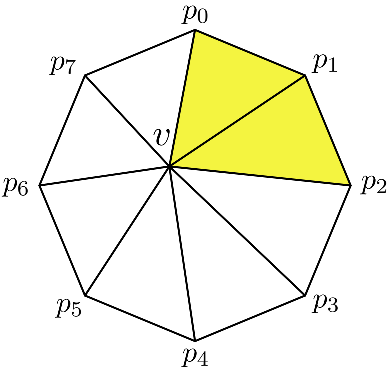
So, v has degree at least 4.

No empty 3-cycles



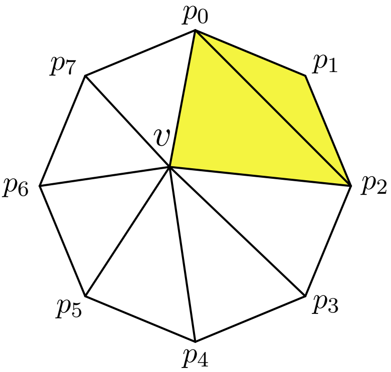
Consider the triangles incident to v (there are at least 4).

No empty 3-cycles



Remove the triangles vp_0p_1 and vp_1p_2, and add vp_0p_2 and $p_0p_1p_2$ (p_0p_2 is not an edge of $\mathcal{P}$, or vp_0p_2 would be an empty 3-cycle).

No empty 3-cycles



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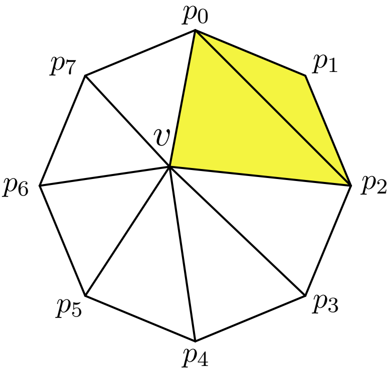
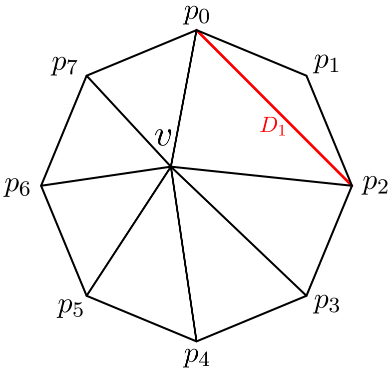
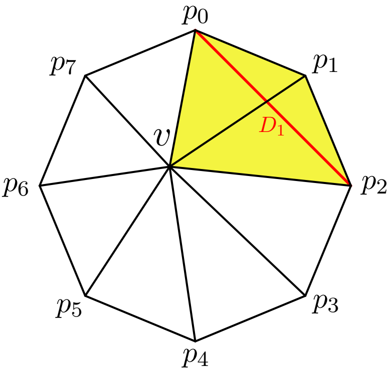


The new polyhedron $\mathcal{P}'$ has the same genus and number of vertices as $\mathcal{P}$, and v has lower degree.

No empty 3-cycles

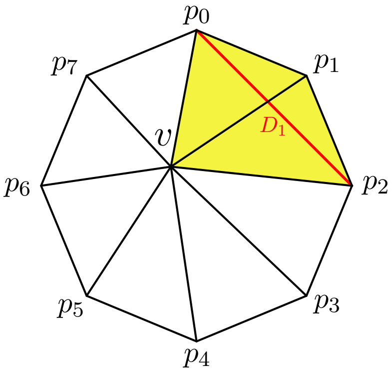


Hence the inductive hypothesis applies to $\mathcal{P}'$, but its edges include $p_0p_2 = D_1$, and its polynomial has coefficients in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2, D_1^2]$.



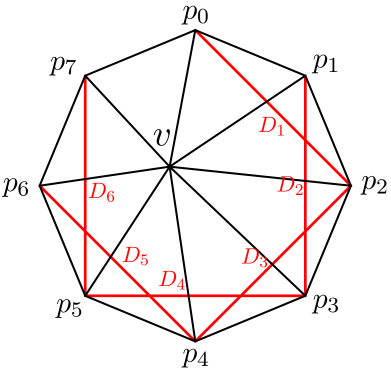
The inductive hypothesis also holds on the tetrahedron $vp_0p_1p_2$, and its polynomial has coefficients in $\mathbb{Q}[\ell_1^2, \cdots, \ell_e^2, D_1^2]$.

No empty 3-cycles



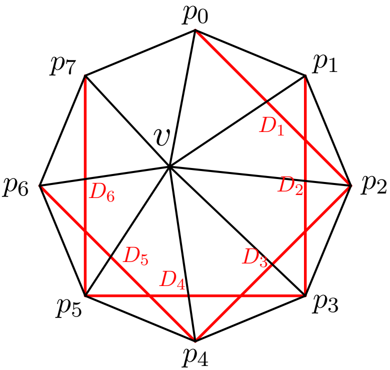
Since the difference between $\mathcal{P}$ and $\mathcal{P}'$ is the tetrahedron $vp_0p_1p_2$, by the Lemma the volume V of $\mathcal{P}$ satisfies $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$.

No empty 3-cycles



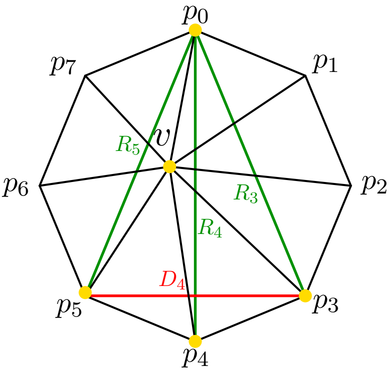
We can repeat the same reasoning with the other edges incident to v, obtaining equations of the form $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_i^2)$.

No empty 3-cycles



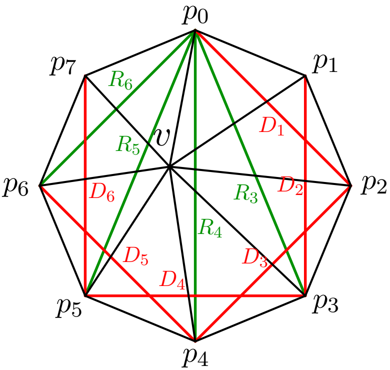
We would like to eliminate the D_i^2's from these polynomial equations. Hence we need more equations involving them.

No empty 3-cycles



The Cayley-Menger determinant applied to v, p_0, p_{i-1}, p_i, p_{i+1} yields an equation of the form $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_i^2, R_{i-1}^2, R_i^2, R_{i+1}^2)$.

No empty 3-cycles



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Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
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- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
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Note that R_1^2 and R_7^2 are in $\ell_1^2, \cdots, \ell_e^2$, so they can be removed.

Eliminating the extra diagonals

We have the following equations:

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- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_2^2, R_2^2, R_3^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_3^2, R_2^2, R_3^2, R_4^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_4^2, R_3^2, R_4^2, R_5^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_5^2, R_4^2, R_5^2, R_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_6^2, R_5^2, R_6^2)$

Note that $R_2^2 = D_1^2$.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, R_3^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_3^2, R_3^2, R_4^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_4^2, R_3^2, R_4^2, R_5^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_5^2, R_4^2, R_5^2, R_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_6^2, R_5^2, R_6^2)$

Apply elimination theory to variable R_6^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, R_3^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_3^2, R_3^2, R_4^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_4^2, R_3^2, R_4^2, R_5^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_5^2, D_6^2, R_4^2, R_5^2)$

Note that the determinant of the Sylvester matrix is a polynomial.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, R_3^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_3^2, R_3^2, R_4^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_4^2, R_3^2, R_4^2, R_5^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_5^2, D_6^2, R_4^2, R_5^2)$

Apply elimination theory to variable R_5^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, R_3^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_3^2, R_3^2, R_4^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_4^2, D_5^2, D_6^2, R_3^2, R_4^2)$

Apply elimination theory to variable R_4^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, R_3^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_3^2, D_4^2, D_5^2, D_6^2, R_3^2)$

Apply elimination theory to variable R_3^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_6^2)$
- $\mathrm{Poly}(\ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, D_3^2, D_4^2, D_5^2, D_6^2)$

Apply elimination theory to variable D_6^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_5^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, D_3^2, D_4^2, D_5^2)$

Apply elimination theory to variable D_5^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_4^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, D_3^2, D_4^2)$

Apply elimination theory to variable D_4^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_3^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2, D_3^2)$

Apply elimination theory to variable D_3^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_2^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2, D_2^2)$

Apply elimination theory to variable D_2^2 in the last two equations.

Eliminating the extra diagonals

We have the following equations:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$
- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2, D_1^2)$

Apply elimination theory to variable D_1^2 in the two equations.

Eliminating the extra diagonals

We have the following equation:

- $\mathrm{Poly}(V, \ell_1^2, \cdots, \ell_e^2)$

This polynomial equation shows that the theorem is valid for $\mathcal{P}$.

One last check

We need to verify that the polynomials we obtain as determinants of the Sylvester matrices are monic in V, and in particular not null!

This can be verified directly by expanding the determinants and keeping track of the coefficients of the leading terms.

Checking it manually is tedious (Sabitov dedicates 12 pages to it), but it is a mechanical manipulation of polynomials that could be carried out by dedicated computer software.