Edge-guarding Orthogonal Polyhedra
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Planar version: Given a polygon, choose a minimum number of vertices that collectively see its whole interior.
Art Gallery Problem

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- **Problem**: Generalize to *orthogonal polyhedra*. 
Terminology

Polyhedra

- genus 0
- genus 1
- genus 2
Terminology

Orthogonal polyhedron

Reflex edge
Guarding polyhedra

- Vertex guards vs. edge guards.
The Art Gallery Problem for vertex guards is unsolvable on some orthogonal polyhedra.

Some points in the central region are invisible to all vertices.
Some orthogonal polyhedra require $\Omega(n^{3/2})$ point guards.
Closed edge guards vs. open edge guards.
**Edge guards**

- Closed edge guards vs. open edge guards.

- **Motivation for open edge guards:** Each illuminated point receives light from a non-degenerate subsegment of a guard.
Closed edge guards vs. open edge guards.

Motivation for open edge guards: Each illuminated point receives light from a non-degenerate subsegment of a guard.

Problem: How much more powerful are closed edge guards?
Closed vs. open edge guards

- Closed edge guards are at least 3 times more powerful.
- No open edge can see more than one red dot.
Closed vs. open edge guards

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  - No open edge can see more than one red dot.

- Is this lower bound tight?
Closed vs. open edge guards

- Each endpoint of a closed edge guard can be replaced by an adjacent open edge.
  - Case analysis on all vertex types.
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- Hence each closed edge guard can be replaced by 3 open edge guards, and our previous bound is tight.
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Our parameters for bounding edge guards in orthogonal polyhedra are the total number of edges $e$ and the number of reflex edges $r$. 
Lower bound

- Asymptotically, $\frac{e}{12}$ edge guards may be necessary.
Asymptotically, $\frac{r}{2}$ reflex edge guards may be necessary.
Observation: Any polyhedron is guarded by the set of its edges.

Upper bound: $e$. 
Observation: Any polyhedron is guarded by the set of its edges.
  - Upper bound: \( e \).

Observation: Any polyhedron is guarded by the set of its reflex edges.
  - Upper bound: \( r \).
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- Upper bound: $e$.

Observation: Any polyhedron is guarded by the set of its reflex edges.

- Upper bound: $r$.

State of the art (Urrutia)

Any orthogonal polyhedron is guardable by $\frac{e}{6}$ closed edge guards.

- Can it be lowered and extended to open edge guards?
Theorem

Any orthogonal polyhedron is guardable by \( \frac{e + r}{12} \) open edge guards.
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Proof.
We select a coordinate axis \( X \) and only place guards on \( X \)-parallel edges.
There are 8 types of \( X \)-parallel edges, and we place guards on the circled ones (\( X \) axis pointing toward the audience):
Improving the upper bound

There are 4 symmetric ways of picking edge types:

\[ \alpha + \beta' + \delta', \]
\[ \gamma + \beta' + \delta', \]
\[ \beta + \alpha' + \gamma', \]
\[ \delta + \alpha' + \gamma'. \]

The sum is

\[ \alpha + \beta + \gamma + \delta + 2\alpha' + 2\beta' + 2\gamma' + 2\delta' = e_x + r_x. \]

Hence, one of the 4 choices picks at most \( \frac{e_x + r_x}{4} \) edges.

By selecting the axis \( X \) that minimizes the sum \( e_x + r_x \), we place at most \( \frac{e + r}{12} \) guards.
Indeed, every $X$-orthogonal section is guarded:

For a given $p$, pick the maximal segment $pq$ and slide it to the left, until it hits a vertex $v$, which corresponds to a selected edge. □
Improving the upper bound

Theorem

For every orthogonal polyhedron of genus $g$,

\[
\frac{1}{6}e + 2g - 2 \leq r \leq \frac{5}{6}e - 2g - 12
\]

holds. Both inequalities are tight for every $g$. 

Corollary

11\:

\[
\frac{1}{6}e - g - 1
\]

open edge guards are sufficient to guard any orthogonal polyhedron.

Corollary

7\:

\[
\frac{5}{6}e - 2g - 12
\]

open edge guards are sufficient to guard any orthogonal polyhedron.
Improving the upper bound

**Theorem**

For every orthogonal polyhedron of genus $g$,

$$\frac{1}{6}e + 2g - 2 \leq r \leq \frac{5}{6}e - 2g - 12$$

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**Corollary**

$$\frac{11}{72}e - \frac{g}{6} - 1$$ open edge guards are sufficient to guard any orthogonal polyhedron.

**Corollary**

$$\frac{7}{12}r - g + 1$$ open edge guards are sufficient to guard any orthogonal polyhedron.
Concluding remarks

- We showed that closed edge guards are 3 times more powerful than open edge guards, for orthogonal polyhedra.

- We lowered the upper bound on the number of edge guards from $\frac{e}{6}$ to $\frac{11}{72}e$, whereas the best known lower bound is $\frac{e}{12}$.

- We gave the new upper bound $\frac{7}{12}r$, whereas the best known lower bound is $\frac{r}{2}$. 
Further research

Conjecture

Any orthogonal polyhedron is guardable by $\frac{e}{12}$ edges and $\frac{r}{2}$ reflex edges.
Conjecture

Any orthogonal polyhedron is guardable by $\frac{e}{12}$ edges and $\frac{r}{2}$ reflex edges.

How to bound the number of guards in terms of $r$, while actually placing them on reflex edges only?

Theorem (O’Rourke)

Any orthogonal prism is guardable by $\left\lceil \frac{r}{2} \right\rceil + 1$ reflex edge guards.
### Further research

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How to bound the number of guards in terms of $r$, while actually placing them on reflex edges only?

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<td>Any orthogonal polyhedron with reflex edges in just two directions is guardable by $\left\lfloor \frac{r}{2} \right\rfloor + 1$ reflex edge guards.</td>
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<td>Any orthogonal polyhedron is guardable by $\left\lceil \frac{2}{3}r \right\rceil$ reflex edge guards.</td>
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- What if we consider polyhedra with faces in 4 different directions?
  - Orthogonal polyhedra come as a subclass.
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**Theorem**

*Any such polyhedron is guardable by $\frac{e+r}{6}$ open edge guards.*
Thank you!